Lecture 5: Quasi-coherent sheaves

Addendum to §3.4

Proposition (Universal properties of $\mathrm{Anim}^{nc}(\mathcal{C})$)
$\mathcal{C}, \mathcal{D}$ additive algebraic categories
$F : \mathrm{Anim}(\mathcal{C}) \longrightarrow \mathrm{Anim}(\mathcal{D})$

1) If F commutes with Ω then it extends uniquely to a functor

$$F^{nc} : \mathrm{Anim}^{nc}(\mathcal{C}) \longrightarrow \mathrm{Anim}^{nc}(\mathcal{D})$$

such that $\Omega^{\infty-n} \circ F^{nc} \simeq F \circ \Omega^{\infty-n}$. $\forall n \geq 0$

[Informally $(X_0, X_1, \dots) \longmapsto (F(X_0), F(X_1), \dots)$.

2) If F $\overset{\text{colimit-preserving}}{\vee}$ commutes with Σ_1 then it extends uniquely to a functor (colimit-preserving)

$$F^{nc} : \mathrm{Anim}^{nc}(\mathcal{C}) \longrightarrow \mathrm{Anim}^{nc}(\mathcal{D})$$

such that $\underline{F^{nc} \circ \Sigma_1^{\infty-n} \simeq \Sigma_1^{\infty-n} \circ F}$ $\forall n \geq 0$
where $\Sigma_1^{\infty-n} : \mathrm{Anim}(\mathcal{C}) \longrightarrow \mathrm{Anim}^{nc}(\mathcal{C})$ is left adjoint to $\Omega^{\infty-n}$.

$\Sigma_1^{\infty} : \mathrm{Anim}(\mathcal{C}) \hookrightarrow \mathrm{Anim}^{nc}(\mathcal{C})$

$\Rightarrow$ F^{nc} preserves connective objects

Proof 1)

$$\begin{tikzcd} \mathrm{Anim}^{\mathrm{uc}}(\mathcal{C}) \arrow[r] \arrow[d, "F^{\mathrm{uc}}"] & \cdots \arrow[r] & \mathrm{Anim}(\mathcal{C}) \arrow[r, "\Omega"] \arrow[d, "F"'] & \mathrm{Anim}(\mathcal{C}) \arrow[d, "F"] \\ \mathrm{Anim}^{\mathrm{uc}}(\mathcal{D}) \arrow[r] & \cdots \arrow[r] & \mathrm{Anim}(\mathcal{D}) \arrow[r, "\Omega"] & \mathrm{Anim}(\mathcal{D}) \end{tikzcd}$$

2) Dually one can prove:

$$\mathrm{Anim}(\mathcal{C}) \xrightarrow{\Sigma} \mathrm{Anim}(\mathcal{C}) \xrightarrow{\Sigma} \mathrm{Anim}(\mathcal{C}) \longrightarrow \cdots \longrightarrow \mathrm{Anim}^{\mathrm{uc}}(\mathcal{C})$$

colimit in the ∞-category of presentable ∞-categories and colimit-limit preserving functors

($\mathrm{Pr}^L \simeq (\mathrm{Pr}^R)^{op}$ limit-preserving functor $\mathrm{Pr}^R \longrightarrow \widehat{\mathrm{Cat}_\infty}$)

Example: $\phi : A \to B$ ring homo.

$\Rightarrow$ $\mathbb{L}\phi^* : D(A) \longrightarrow D(B)$ which restricts to
$\mathbb{L}\phi^* : D(A)_{\geq 0} \longrightarrow D(B)_{\geq 0}$

and $\phi_* : D(B) \longrightarrow D(A)$ which restricts to
$\phi_* : D(B)_{\geq 0} \longrightarrow D(A)_{\geq 0}$.

5.1 Descent for animated modules

Reminder: Zariski topology on CRing^{op} is the Grothendieck topology generated by families

$$(A \xrightarrow{\phi_i} A_i)_i$$

where ϕ_i: flat epimorphism $(A_i \otimes_A A_i \xrightarrow{\sim} A_i)$ of finite pres. such that $A \to \prod_i A_i$ is faithfully flat.

- Equivalently: finite families $(A \to A[f_i^{-1}])_i$ here $f_i \in A$ jointly generate the unit ideal of A
- Under the equivalence $\mathrm{CRing}^{op} \simeq \mathrm{Sch}^{aff}$ this is the big affine Zariski site.

Theorem: The functors

$$D_{\geq 0}, D : \mathrm{CRing} \longrightarrow \infty\text{-Cat}$$

$$A \mapsto D(A)_{\geq 0}, \quad \phi \mapsto L\phi^*$$
$$A \mapsto D(A), \quad \phi \mapsto L\phi^*$$

satisfy Zariski descent.

- In particular for every family $(A \to A_i)$, generating a Zar covering sieve, there is a limit diagram

$$D(A) \longrightarrow \prod_i D(A_i) \rightrightarrows \prod_{i,j} D(A_i \otimes_A A_j) \substack{\rightarrow\\\rightarrow\\\rightarrow}$$

in the ∞-category of ∞-categories

5.2. Quasi-coherent sheaves on affine schemes

Theorem. $X = \mathrm{Spec}(A)$ affine scheme
There exists a unique Zariski sheaf of ∞-categories on the small Zariski site X_{Zar}

$$\mathcal{D} \quad (X_{Zar})^{op} \longrightarrow \infty\text{-}Cat$$

whose values on elementary opens $U(f)$ are $\mathcal{D}(U(f)) \simeq \mathcal{D}(A[f^{-1}])$
Moreover: any affine open $U = \mathrm{Spec}(B) \subseteq X$

$$\mathcal{D}(U) \simeq \mathcal{D}(B).$$

Rmk: $\Longrightarrow \exists!\ \mathcal{O}_X \in \mathcal{D}(X)$ whose restriction to any $U(f) \subseteq X$ is $A[f^{-1}] \in \mathcal{D}(A[f^{-1}]) \cong \mathcal{D}(U(f))$.

This is the structure sheaf of X.
(Happens to be discrete)

Proof of Thm from §5.1.

Claim $\mathcal{D} : (Sch^{aff})^{op} \to \infty\text{-}Cat$, $\mathrm{Spec}(A) \mapsto \mathcal{D}(A)$ is a Zar sheaf

$\forall\ X \in Sch^{aff}$ $(U_\alpha \to X)_\alpha$ Zar. covering in Sch^{aff}

$$\mathcal{D}(X) \xrightarrow{\sim} \mathrm{Tot}(\mathcal{D}(\check{C}(U_\bullet/X)_\bullet)) \quad \text{equivalence.}$$

Note: U_α are all affine, so this is a special case of the previous theorem. ∎

Remark. Descent for D follows from $D_{\geq 0}$
$\uparrow$ limit of $D_{\geq 0}$'s

The sheaf condition only involves limits
Limits commute with limits.

Lemma (fpqc conservativity).
$(\phi_\alpha : A \to B_\alpha)_\alpha$ finite family of flat ring homos
such that $A \to B := \prod_\alpha B_\alpha$ is faithfully flat

Then the family of functors $\phi_\alpha^* : D(A)_{\geq 0} \to D(B_\alpha)_{\geq 0}$
is jointly conservative

Proof

- Note $D(B)_{\geq 0} \simeq \prod_\alpha D(B_\alpha)_{\geq 0}$
  Therefore we may as well assume that the family consists of a single faithfully flat map $\phi : A \to B$.

- **Claim:** $\phi^*(M) \simeq 0 \implies M \simeq 0 \quad (M \in D(A)_{\geq 0})$

- **Note.** $M \simeq 0 \iff \pi_i(M) = 0 \quad \forall i \geq 0$
  ($M \mapsto M^\circ$ is conservative)

  $\iff \pi_i(M) \otimes_A B \simeq 0 \quad \forall i \quad (A \to B$ faithfully flat$)$

  $\iff \pi_i(M \otimes^L_A B) \simeq 0 \quad \forall i \quad$ (because $A \to B$ flat)

  $\iff L\phi^*(M) := M \otimes^L_A B \simeq 0$. ∎

Proof of Thm (§5 2)

Apply last lecture. Suff to show

- Claim: $U, V \subseteq X$ affine opens s.th. $U \cup V$ affine.
  Then

$$\begin{tikzcd} D(U\cup V)_{\geq 0} \arrow[r] \arrow[d] & D(U)_{\geq 0} \arrow[d] \\ D(V)_{\geq 0} \arrow[r] & D(U \cap V)_{\geq 0} \end{tikzcd}$$

cartesian square of ∞-categories.

(Take $\mathcal{U}_{aff}(X) \subseteq \mathcal{U}(X)$ affine opens. This is an $\cap$-closed basis since X affine.)

- WLOG: $X = U \cup V$.

- $A := \Gamma(X, \mathcal{O}_X)$ $\quad A_1 = \Gamma(U, \mathcal{O}_U)$ $\quad A_2 = \Gamma(V, \mathcal{O}_V)$
  $A_{12} = \Gamma(U \cap V, \mathcal{O}_{U\cap V})$

Want: $F : D(A)_{\geq 0} \xrightarrow{\sim} D(A_1)_{\geq 0} \times_{D(A_{12})_{\geq 0}} D(A_2)_{\geq 0}$

$G : D(A_1)_{\geq 0} \times_{D(A_{12})_{\geq 0}} D(A_2)_{\geq 0} \longrightarrow D(A)_{\geq 0}$ right adjoint

$(M_1, M_2, M_1 \otimes_{A_1} A_{12} \simeq M_2 \otimes_{A_2} A_{12}) \longmapsto M_1 \times_{M_{12}} M_2$

$M_1 \in D(A_1)_{\geq 0}$, $M_2 \in D(A_2)_{\geq 0}$

Check unit: $\mathrm{id} \xrightarrow{\sim} GF$

$\forall M \in D(A)_{\geq 0}$ $\quad M \xrightarrow{\sim} M_1 \times_M M_2$ $\quad M_? := M \otimes_A A_?$

Apply fpqc conservativity: (derived) extension of scalars along $A \to A_1$, $A \to A_2$ is jointly conservative. $(A \to A_1, A \to A_2)$ is a fin. family of flat homos. jointly faithfully flat

Want: $M \otimes_A A_i \longrightarrow \left(M_1 \underset{M_{12}}{\times} M_2\right) \otimes_A A_i$ iso $\forall i \in \{1,2\}$

$$\simeq (M_1 \otimes_A A_i) \underset{(M_{12} \otimes_A A_i)}{\times} (M_2 \otimes A_i)$$

Wlog $i = 1$

$$\simeq (M \otimes_A \underbrace{A_1 \otimes A_1}_{A}) \underset{(M \otimes_A \underbrace{A_{12} \otimes A_1}_{A})}{\times} (M \otimes_A \underbrace{A_2 \otimes A_1}_{A})$$

$A_1 \otimes_A A_1 \simeq A_1$

$A_{12} \simeq A_1 \otimes_{A_0} A_2$

$\simeq A_1 \otimes_A A_2$

$$\simeq M \otimes_A A_1$$

Now it suffices to show G is conservative

Left as exercise. ∎

$\Longrightarrow$ Right Kan extension along $\mathcal{U}_{aff}(X) \subseteq \mathcal{U}(X)$

$\rightsquigarrow$ sheaf on X

§ 3. Quasi-coherent sheaves on schemes

Theorem: 1) There is a unique Zar sheaf

$D : (Sch)^{op} \longrightarrow \infty\text{-Cat}$

which extends $D : (Sch^{aff})^{op} \to \infty\text{-Cat}$.

- Recall: Sch big Zariski site
  topology generated by $(U_i \longrightarrow X)_i$ open immersions
  $\coprod U_i \twoheadrightarrow X$ jointly surj

2) Moreover $D(X) \simeq \varprojlim_{(S, S \to X)} D(A)$ $\quad X \in Sch$

over pairs $(S, S \to X)$ where $\underset{\mathrm{Spec}(A)}{\overset{\shortparallel}{S}}$ affine and $S \to X$ is a morphism

Proof. • There is an equivalence

$$Shv^{\mathcal{V}}_{Zar}(Sch) \xrightarrow{\sim} Shv^{\mathcal{V}}_{Zar}(Sch^{aff}) \qquad \mathcal{V}\ \infty\text{-cat with limits}$$

$Sch^{aff} \subseteq Sch^{sep}$ and $Sch^{sep} \subseteq Sch$ are bases which are $\cap$-closed.

- $D = RKE$ of $D : (Sch^{aff})^{op} \to \infty\text{-Cat}$ $\blacksquare$

Rmk: No analogue for triangulated categories.

$$\begin{aligned} hD : (Sch)^{op} &\to TriCat \to Cat \\ X &\mapsto hD(X) \end{aligned}$$ do not satisfy descent.

$D(\mathbb{P}^1) \xrightarrow{\sim} D(\mathbb{A}^1) \times_{D(\mathbb{A}^1 - 0)} D(\mathbb{A}^1)$ fails at hD level.

Cor: $X \in Sch$ $(X_{Zar})^{op} \xrightarrow{D} \infty\text{-}Cat$ is Zar sheaf.

Proof by restriction. ∎

Proposition $X \in Sch$ The Zariski sheaf $D : (X_{Zar})^{op} \to \infty\text{-}Cat$ is the right Kan extension of its restriction to $\mathcal{U}_{aff}(X)$

In particular: $\forall U \subseteq X$ open $D(U) \xrightarrow{\sim} \varprojlim_{\substack{V \subseteq U \\ \text{aff opens}}} D(V)$

$\Rightarrow$ D is the unique Zar sheaf on X_{Zar} whose values on affine opens $U = \mathrm{Spec}(A) \subseteq X$ are $D(U) \simeq D(A)$

Proof: $\mathcal{U}_{sep}(X) \subseteq \mathcal{U}(X)$ separated opens form a $\cap$-closed basis

$\mathcal{U}_{aff}(X) \subseteq \mathcal{U}_{sep}(X)$ form a $\cap$-closed basis

Def • A <u>quasi-coherent complex</u> on a scheme X is an object $\mathcal{F} \in D(X)$

• A <u>quasi-coherent animated sheaf</u> / <u>connective qcoh. complex</u> on X is an object ~~[illegible]~~ $\mathcal{F} \in D(X)_{\geq 0}$

Ex. • $X = \mathrm{Spec}(A)$ affine $\qquad D(X) \simeq D(A)$

• X scheme $\qquad D(X) \xrightarrow{\sim} \varprojlim_{U \subseteq X} D(U)$ $\quad U$ affine opens

A qcoh complex on X amounts to:

• $\forall\, U = \mathrm{Spec}(A) \subseteq X$ a noncomm. animated A-module $\mathcal{F}_U \in D(U) \simeq D(A)$

• $\forall\, V \subseteq U \subseteq X$ inclusion of affine opens an isomor $\mathcal{F}_U|_V \simeq \mathcal{F}_V$.

• homotopy coherent system of compatibilities between these isos.

$\mathcal{F} \in D(X)$ is <u>connective</u> $\iff \forall\, \overset{\mathrm{Spec}(A)}{\overset{\shortparallel}{U}} \subseteq X$ affine open

$\Gamma(U, \mathcal{F}) \in D(A)$ is a <u>connective</u> animated module ($\in D(A)_{\geq 0}$).

$\mathcal{F} \in D(X)^{\heartsuit}$ <u>discrete</u> $\iff \forall\, U = \mathrm{Spec}(A) \subseteq X$ $\Gamma(U, \mathcal{F}) \in D(A)$ is discrete

<u>**Cor.**</u> • $X \mapsto D(X)^{\heartsuit}$ Zariski sheaf of categories determined by $D(\mathrm{Spec}(A))^{\heartsuit} \simeq \mathrm{Mod}_A$ $\forall A \in \mathrm{CRing}$

• $\Longrightarrow$ $D(X)^{\heartsuit} \simeq QCoh(X)$ = ab. cat of qcoh $\mathcal{O}_X$-modules

<u>Rmk</u> $D(X)$ is a stable ∞-cat $\forall X \in Sch$

(Limits of stable ∞-cat's are stable).

$D(X)_{\geq 0}$ prestable