6. Direct image functor

Def: (Inverse image functor)

$f : X \to Y$ morphism of schemes $\qquad$ $D \; (Sch)^{op} \to \infty\text{-Cat}$

$D(f) : D(Y) \longrightarrow D(X)$ $\qquad$ inverse image $\quad$ denoted $\mathbb{L}f^* := D(f)$

$$\begin{tikzcd} D(Y)_{\geq 0} \arrow[r, "\mathbb{L}f^*"] \arrow[d, phantom, "\cap" rotate=90] & D(X)_{\geq 0} \arrow[d, phantom, "\cap" rotate=90] \\ D(Y) \arrow[r, "\mathbb{L}f^*"] & D(X) \end{tikzcd}$$

Ex: • $f : X \to Y$ morphism of affine schemes $\quad X = \mathrm{Spec}(B)$, $Y = \mathrm{Spec}(A)$

Corresp. to $\phi : A \to B$

$$\begin{array}{ccccc} \mathbb{L}f^* = \mathbb{L}\phi^* : & D(Y) & \to & D(X) \\ & \wr\! \| & & \wr\! \| \\ & D(A) & \to & D(B). \end{array}$$

• $j : U \hookrightarrow X$ open affine subscheme

$$\begin{tikzcd} \mathcal{F} = (\mathcal{F}_V)_V \in D(X) \arrow[r, "\sim"] \arrow[d, mapsto] \arrow[dr, "j^*"] & \lim\limits_{\substack{\longleftarrow \\ V \subseteq X \\ \text{affine}}} D(V) \arrow[d] \\ \mathcal{F}_U \in & D(U) \end{tikzcd}$$

commutes.

Example: f flat $f : X \to Y$

$$\Rightarrow \mathbb{L}f^{*} : D(Y) \longrightarrow D(X) \quad \text{preserves discrete objects}$$

$$\begin{array}{ccc} D(Y) & \longrightarrow & D(X) \\ \cup & & \cup \\ D(Y)^{\heartsuit} & \xrightarrow{f^{*}} & D(X)^{\heartsuit} \end{array}$$

(Follows by descent from affine case.)

Corollary ("Internal descent"):
X scheme, $\mathcal{F} \in D(X)$

Notation: $\Gamma(U, \mathcal{F}) := \mathrm{Maps}_{D(U)}(\mathcal{O}_U, \underset{\substack{\| \\ j^*(\mathcal{F})}}{\mathcal{F}_U}) \in \mathrm{Anim}$
$U \subseteq X$ open

Claim. $\Gamma(-, \mathcal{F})$ is a sheaf of anima on X_{Zar}

$\Rightarrow$ (X qcqs)

$$\begin{tikzcd} \Gamma(U \cup V, \mathcal{F}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \Gamma(U, \mathcal{F}) \arrow[d] \\ \Gamma(V, \mathcal{F}) \arrow[r] & \Gamma(U \cap V, \mathcal{F}) \end{tikzcd}$$

cartesian square $\forall U, V \subseteq X$ qc opens

Proof: Note that for any $\mathcal{G}, \mathcal{G}' \in D(U \cup V)$

$$\begin{tikzcd} \mathrm{Maps}_{D(U \cup V)}(\mathcal{G}, \mathcal{G}') \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Maps}_{D(U)}(\mathcal{G}|_U, \mathcal{G}'|_U) \arrow[d] \\ \mathrm{Maps}_{D(V)}(\mathcal{G}|_V, \mathcal{G}'|_V) \arrow[r] & \mathrm{Maps}_{D(U \cap V)}(\mathcal{G}|_{U \cap V}, \mathcal{G}'|_{U \cap V}) \end{tikzcd}$$

Cartesian square

This follows from the fact that the square

$$\begin{tikzcd} D(U \cup V) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & D(U) \arrow[d] \\ D(V) \arrow[r] & D(U \cap V) \end{tikzcd}$$

is cartesian

because "formation of mapping spaces of ∞-categories commutes with limits".

Apply this with $\mathcal{G} = \mathcal{O}_{U \cup V}$ $\mathcal{G}' = \mathcal{F}|_{U \cup V}$. ■

6.1 Direct image

Def.: $f: X \to Y$ $\quad \mathbb{R}f_*: D(X) \to D(Y)$ direct image is the right adjoint of $\mathbb{L}f^*$

Ex: • $X = \mathrm{Spec}(B)$, $Y = \mathrm{Spec}(A)$ $\quad \phi: A \to B$

$\mathbb{R}f_* = \phi_*: D(A) \to D(B)$

$\Rightarrow$ preserves connective and discrete objects (write $f_* := \mathbb{R}f_*$)

Warning: $\mathbb{R}f_*$ typically does not preserve connectivity for non-affine morphisms

<u>Theorem</u>: $f: X \to Y$ quasi-compact quasi-sep morphism

1) $\mathbb{R}f_*$ commutes with colimits

2) Base change formula:

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner"] & Y' \arrow[d, "q"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

$\mathbb{L}q^*\mathbb{R}f_* \longrightarrow \mathbb{R}g_*\mathbb{L}p^*$

If p, q are flat (more generally: Tor-independent squares) then this is an iso.

3) Projection formula: $\mathcal{F} \in D(X)$, $\mathcal{G} \in D(Y)$

$\mathbb{R}f_*(\mathcal{F}) \overset{\mathbb{L}}{\otimes} \mathcal{G} \xrightarrow{\sim} \mathbb{R}f_*(\mathcal{F} \overset{\mathbb{L}}{\otimes} \mathbb{L}f^*(\mathcal{G}))$ canonical iso

<u>Rmk</u>.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "f"] \arrow[d, "p"'] & \mathcal{C}' \arrow[d, "q"] \\ \mathcal{D} \arrow[r, "g"'] & \mathcal{D}' \end{tikzcd}$$

$\boxed{g \circ p = q \circ f}$

commutative square of ∞-categories

Assume p, q have right adjoints p^R and q^R resp.

Then the square

$$\begin{tikzcd} \mathcal{D} \arrow[r, "g"] \arrow[d, "p^R"'] & \mathcal{D}' \arrow[d, "q^R"] \\ \mathcal{C} \arrow[r, "f"'] & \mathcal{C}' \end{tikzcd}$$

$f \circ p^R \xrightarrow{\text{unit}} q^R \circ \underbrace{q \circ f}_{} \circ p^R \simeq q^R \circ g \circ \underbrace{p \circ p^R}_{} \xrightarrow{\text{counit}} q^R \circ g$

commutes up to a natural transformation.

We say that the (original) square is <u>vertically right-adjointable</u> if this natural transformation is an iso.

Note: The base change formula says that

$$\begin{tikzcd} D(Y) \arrow[r, "\mathbb{L}q^*"] \arrow[d, "\mathbb{L}f^*"'] & D(Y') \arrow[d, "\mathbb{L}g^*"] \\ D(X) \arrow[r, "\mathbb{L}p^*"'] & D(X') \end{tikzcd}$$

is vertically right-adjointable

<u>Rmk</u>: $\mathbb{L}f^* : D(Y) \longrightarrow D(X)$ symmetric monoidal

$\Rightarrow$ $D(X)$ has a canonical $D(Y)$-module structure.

Projection formula for $\mathbb{R}f_*$ says that $\mathbb{R}f_*$ is $D(Y)$-linear (morphism of $D(Y)$-module ∞-categories)

<u>Lemma</u>: $\left(\Phi_i : \mathcal{D}_i \to \mathcal{C}_i\right)_{i \in I}$ diagram in $\mathrm{Fun}(\Delta^1, \infty\text{-Cat})$, I (∞-)category

Suppose that each square

$$\begin{tikzcd} \mathcal{D}_i \arrow[r] \arrow[d, "\Phi_i"'] & \mathcal{D}_j \arrow[d, "\Phi_j"] \\ \mathcal{C}_i \arrow[r] & \mathcal{C}_j \end{tikzcd}$$

$\forall\ i \to j$ in I is vertically right adjointable.

Then: Consider induced functor

$$\begin{tikzcd} \mathcal{D} \simeq \varprojlim_i \mathcal{D}_i \arrow[d, "\Phi"'] \\ \mathcal{C} \simeq \varprojlim_i \mathcal{C}_i \end{tikzcd}$$

The square
$$\begin{tikzcd} \mathcal{D} \arrow[r, "pr_i"] \arrow[d, "\Phi"'] & \mathcal{D}_i \arrow[d, "\Phi_i"] \\ \mathcal{C} \arrow[r] & \mathcal{C}_i \end{tikzcd}$$

is vertically right-adjointable $\forall i \in I$.

<u>Proof of base change in affine case.</u>

By the lemma, reduce to <u>animated</u> modules (connective complexes)

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "\psi"'] & B \arrow[d, "\psi'"] \\ A' \arrow[r, "\phi'"'] & B' \end{tikzcd}$$

$B' \simeq B \otimes_A A' \simeq B \overset{L}{\otimes}_A A'$

$$\mathbb{L}\psi^* \phi_* \xrightarrow{?} \phi'_* \mathbb{L}(\psi')^*$$

$$N \overset{L}{\otimes}_A A' \longrightarrow N \overset{L}{\otimes}_B B' \qquad N \in D(B)_{\geq 0}$$

But $N \overset{L}{\otimes}_B B' \simeq N \overset{L}{\otimes}_B B \overset{L}{\otimes}_A A' \simeq N \overset{L}{\otimes}_A A'$ ∎

6.2: Descent for the direct image functor

Rmk: X scheme, $\mathcal{F} \in D(U \cup V)$

$U, V \subseteq X$ opens, $j_U : U \hookrightarrow X$ inclusion $(V, U\cap V)$

Then
$$j_{U\cup V,*}(\mathcal{F}) \xrightarrow{\sim} j_{U,*}(\mathcal{F}_U) \times_{j_{U\cap V,*}(\mathcal{F}_{U\cap V})} j_{V,*}(\mathcal{F}_V).$$

Indeed this follows from the equivalence

$$(j_U^*, j_V^*) : D(U \cup V) \xrightarrow{\sim} D(U) \times_{D(U\cap V)} D(V)$$

Right adjoint: $(\underset{D(U)}{\mathcal{F}}, \underset{D(V)}{\mathcal{G}}, \mathcal{F}|_{U\cap V} \simeq \mathcal{G}|_{U\cap V}) \longmapsto j_{U,*}(\mathcal{F}) \times_{j_{U\cap V,*}(\mathcal{H})} j_{V,*}(\mathcal{G})$, where $\mathcal{F}|_{U\cap V} \cong \mathcal{H}$, $D(U\cap V) \ni \mathcal{H}$

The map in question is the unit of the adjunction.

Cor: $f : X \to Y$, $X = U \cup V$ open cover

$\mathcal{F} \in D(X)$, $f_U = f|_U$, etc

$$f_*(\mathcal{F}) \xrightarrow{\sim} f_{U,*}(\mathcal{F}_U) \times_{f_{U\cap V,*}(\mathcal{F}_{U\cap V})} f_{V,*}(\mathcal{F}_V)$$

Proof: Apply f_* to previous isomorphism.

6.3 Sketch of proof of base change formula

Case 1: $j : U \hookrightarrow X$ open immersion, $U \subseteq X$ quasi-compact

$f : X' \to X$ morphism of affine schemes

$$\begin{tikzcd} U' \arrow[r, hook, "j'"] \arrow[d, "f_U"'] & X' \arrow[d, "f"] \\ U \arrow[r, hook, "j"] & X \end{tikzcd} \qquad f^* j_* \xrightarrow{\sim} j'_* f_U^* : D(U) \to D(X')$$

- $U = V \cup W$ and the claim holds for V and W

$\Rightarrow$ then it holds for U

Follows from "descent for j_*" (§6.2).

- U qc $\Rightarrow$ $U = \bigcup_i U(f_i)$ finite union of elementary opens

Conclude by induction.

Case 2:

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "q"'] & Y' \arrow[d, "p"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Y, Y' affines

$f : X \to Y$ quasi-compact

Argue similarly by induction on an affine open cover of X, using "descent for f_*" (§6.2).

General case: Use "adjointability of limits" Lemma and descent to reduce to the case where Y, Y' affine.

6.4: Direct image along open immersions

Corollary: $U \subseteq X$ quasi-compact open
$j : U \hookrightarrow X$
Then the functor

$$\mathbb{R}j_* : D(U) \longrightarrow D(X)$$

is fully faithful.

- Equivalently, $j^* \mathbb{R}j_* \xrightarrow{\text{counit}} \mathrm{id} : D(U) \to D(U)$ is an isomorphism.

Proof:
$$\begin{tikzcd} U \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner"] & U \arrow[d, hook, "j"] \\ U \arrow[r, hook, "j"'] & X \end{tikzcd}$$
Cartesian square
j flat

$\Rightarrow$ Base change formula. ∎

Warning: $Z \subseteq X$ closed subscheme $i : Z \hookrightarrow X$

$$\begin{tikzcd} Z \arrow[r, equal] \arrow[d, equal] & Z \arrow[d, hook, "i"] \\ Z \arrow[r, hook, "i"] & X \end{tikzcd}$$
not Tor-indep

$\mathbb{R}i_*$ is typically not fully faithful

even though $i_* : \mathrm{Qcoh}(Z) \longrightarrow \mathrm{Qcoh}(X)$ is fully faithful.