Exercise sheet 1

The minimum passing average is 20 points per sheet.

1. *10 points.* Let X be a topological space. Let U and V be open subsets such that $\mathrm{X} = \mathrm{U} \cup \mathrm{V}$. Consider the commutative square of chain complexes

$$\begin{tikzcd} \mathrm{C}_*(\mathrm{U} \cap \mathrm{V}) \arrow[r, "i_{1,*}"] \arrow[d, "i_{0,*}"] & \mathrm{C}_*(\mathrm{V}) \arrow[d, "j_{1,*}"] \\ \mathrm{C}_*(\mathrm{U}) \arrow[r, "j_{0,*}"] & \mathrm{C}_*(\mathrm{X}) \end{tikzcd}$$

where the homomorphisms are pushforwards along the inclusions. Give a counterexample to show that this square is generally not a pushout in the category of chain complexes, or equivalently that the sequence

$$0 \to \mathrm{C}_*(\mathrm{U} \cap \mathrm{V}) \xrightarrow{\begin{bmatrix} i_{1,*} \\ i_{2,*} \end{bmatrix}} \mathrm{C}_*(\mathrm{U}) \oplus \mathrm{C}_*(\mathrm{V}) \xrightarrow{j_{0,*} - j_{1,*}} \mathrm{C}_*(\mathrm{X}) \to 0$$

is not exact.

2. *5 points.* Let F and G be functors $\mathcal{C} \rightrightarrows \mathcal{D}$ between two ordinary categories. Show that natural transformations $\mathrm{F} \Rightarrow \mathrm{G}$ are in bijection with morphisms of simplicial sets $\varphi : \Delta^1 \times \mathrm{N}(\mathcal{C}) \to \mathrm{N}(\mathcal{D})$ such that $d^0(\varphi) = \mathrm{F}$ and $d^1(\varphi) = \mathrm{G}$.

3. *15 points.* Show that a simplicial set X is the nerve of a category (resp. nerve of a groupoid) if and only if it has the lifting property for inclusions $\Lambda^n_i \hookrightarrow \Delta^n$, for $0 < i < n$ (resp. for $0 \leqslant i \leqslant n$). Here Λ^n_i is the subset of Δ^n obtained by removing the interior and the face opposite the ith vertex; see [Cis, 1.4.5.3].

4. *5 points.* Show that the assignment $\mathrm{X} \mapsto c(\mathrm{X})$, sending a set to the associated constant simplicial set, is fully faithful. Show that c admits a left adjoint, given by $\mathrm{X} \mapsto \pi_0(\mathrm{X}) := \mathrm{Coeq}(\mathrm{X}_1 \rightrightarrows \mathrm{X}_0)$, where the arrows are the face maps d^0, d^1.

5. *15 points.* Show that the functor $\mathrm{N} : \mathrm{Cat} \to \mathrm{sSet}$ determined by the nerve is fully faithful.

6. *5 points.* Let X be a topological space and $x \in \mathrm{X}$ a point. Consider the homotopy pullback

$$\begin{tikzcd} \mathrm{pt} \times_{\mathrm{X}} \mathrm{pt} \arrow[r] \arrow[d] & \mathrm{pt} \arrow[d, "x"] \\ \mathrm{pt} \arrow[r, "x"] & \mathrm{X} \end{tikzcd}$$

in the ∞-category **H**. Describe the points of the type $\mathrm{pt} \times_{\mathrm{X}} \mathrm{pt}$.

7. *10 points.* Let $\mathcal{C}$ be a stable ∞-category and suppose we have an exact triangle

$$\mathrm{X} \xrightarrow{f} \mathrm{Y} \xrightarrow{g} \mathrm{Z}$$

in $\mathcal{C}$. Construct the *boundary map* $\partial : \mathrm{Z} \to \mathrm{X}[1]$, and show that

$$\mathrm{Z}[-1] \xrightarrow{\partial} \mathrm{X} \xrightarrow{f} \mathrm{Y} \quad \text{and} \quad \mathrm{Y} \xrightarrow{g} \mathrm{Z} \xrightarrow{\partial} \mathrm{X}[1]$$

are exact triangles in $\mathcal{C}$.

8. *20 points.* Let $\mathcal{C}$ be a stable ∞-category. Let $f : \mathrm{X}_{0,1} \to \mathrm{X}_{1,1}$ and $g : \mathrm{X}_{1,0} \to \mathrm{X}_{1,1}$ be morphisms in $\mathcal{C}$, and denote by P the pullback $\mathrm{X}_{0,1} \times_{\mathrm{X}_{1,1}} \mathrm{X}_{1,0}$. Show that there is a canonical isomorphism
$$\mathrm{P} \simeq \mathrm{Fib}(f - g : \mathrm{X}_{0,1} \oplus \mathrm{X}_{1,0} \to \mathrm{X}_{1,1})$$
in $\mathcal{C}$. (Comparing the *points* of the two types will earn 5 points.)

9. *20 points.* State and prove the dual statement of the previous exercise. (Doing both exercises 8 and 9 will earn 30 points total.)

References

[Cis] D.-C. Cisinski, *Higher categories and homotopical algebra.*