Exercise sheet 12

The minimum passing average is 20 points per sheet.

1. *10 points.* Let $X \in \mathrm{Sch}^{\mathrm{ft}}_{\mathbf{C}}$ and $f : X \to \mathbf{A}^1$ a regular function. If X is smooth, show that $\phi_f(\underline{R}_X) \in \mathrm{Shv}(X_0)$ is supported on the critical points of f. I.e., its supported is contained in $\mathrm{Crit}(f) \cap X_0$, where $\mathrm{Crit}(f) := \{x \in X \mid df_x = 0\}$.

2. *5 points.* Use the contraction lemma to give another solution of Sheet 8, Exercise 3(3).

3. *10 points.* Let $X \in \mathrm{Sch}^{\mathrm{ft}}_{\mathbf{C}}$ and suppose given a $\mathbf{C}^*$-action on X. Let $\mathcal{F} \in \mathrm{Shv}_c(X)$ be a constructible sheaf. Show that $\mathcal{F}$ is monodromic if and only if $\mathcal{F}$ is conic.

   (Hint: Reduce to the assertion that if $\mathcal{F} \in \mathrm{Shv}_c(\mathbf{G}_m)$ is conic, then it is locally constant. Indeed, let U be the open locus where $\mathcal{F}$ is locally constant; since $\mathcal{F}$ is constructible, U is the complement of a finite discrete set of points. Now apply Sheet 11, Exercise 5 to each of those points.)

4. *15 points.* Suppose $R = k$ is a field. The *Euler characteristic* of a perfect complex $K \in \mathbf{D}(k)$ is defined by $\chi(K) := \sum_{i \in \mathbf{Z}} (-1)^i \dim_k \mathrm{H}_i(K)$.

   Let $X \in \mathrm{Sch}^{\mathrm{ft}}_{\mathbf{C}}$ and $\mathcal{F} \in \mathrm{Shv}_c(X; k)$ a constructible k-linear sheaf. The *Euler characteristic* of $\mathcal{F}$ is the Euler characteristic of the complex $\Gamma(X, \mathcal{F}) \in \mathbf{D}(k)$. (Note that the latter is perfect because $\mathcal{F}$ is constructible, and hence $a_*(\mathcal{F})$ is constructible where $a : X \to \mathrm{Spec}(\mathbf{C})$.)

   Suppose X is a smooth irreducible curve. Show that

   $$\chi(X, \mathcal{F}) = \chi(X, \underline{R}_X) \cdot \chi(\eta^*\mathcal{F}) - \sum_{x \in X} \chi(\phi_{t_x}(\mathcal{F}))$$

   where $\eta^*\mathcal{F} \in \mathrm{Shv}(\{\eta\}; k) \simeq \mathbf{D}(k)$ is the stalk of $\mathcal{F}$ at the generic point $\eta \in X$, and for every $x \in X$, t_x is a local coordinate at x (see Sheet 11).