Exercise sheet 2

The minimum passing average is 20 points per sheet.

1. *10 points.* Let $\mathrm{K}_\bullet \in \mathrm{Ch}(\mathrm{R})$ be a chain complex of R-modules, representing an object $\mathrm{K} \in \mathbf{D}(\mathrm{R})$. Suppose $\mathrm{K}_\bullet$ is bounded, i.e. there exist integers $a < b$ such that $\mathrm{K}_i = 0$ for all $i \notin [a, b]$. Show that K can be described by starting with the object $\mathrm{Q}_a := \mathrm{K}_a$, taking the cofibre $\mathrm{Q}_{a+1} := \mathrm{Cofib}(d : \mathrm{K}_{a+1} \to \mathrm{K}_a)$, and iterating inductively until one obtains $\mathrm{K} \simeq \mathrm{Q}_b$.

2. *10 points.* Let $\mathrm{K}_\bullet \in \mathrm{Ch}(\mathrm{R})$ be an (unbounded) chain complex of R-modules, representing an object $\mathrm{K} \in \mathbf{D}(\mathrm{R})$. For each $n \geqslant 0$, denote by $\mathrm{K}(n)_\bullet$ the truncated complex concentrated in degrees $[-n, n]$: $\mathrm{K}(n)_i = \mathrm{K}_i$ if $i \in [-n, n]$ and $\mathrm{K}(n)_i = 0$ otherwise. Show that there are morphisms $\mathrm{K}(m)_\bullet \to \mathrm{K}(n)_\bullet$ whenever $m \leqslant n$, and the colimit $\varinjlim_n \mathrm{K}(n)_\bullet$, taken in $\mathbf{D}(\mathrm{R})$, is canonically isomorphic to K. You may assume the result of the next exercise.

3. *10 points.* Use the fact that filtered colimits in $\mathrm{Ch}(\mathrm{R})$ preserve quasi-isomorphisms to show that if $\{(\mathrm{K}_\alpha)_\bullet\}_\alpha$ is a filtered diagram in $\mathrm{Ch}(\mathrm{R})$, representing a filtered diagram $\{\mathrm{K}_\alpha\}_\alpha$ in $\mathbf{D}(\mathrm{R})$, then the colimit in $\mathrm{Ch}(\mathrm{R})$ is also the colimit $\varinjlim_\alpha \mathrm{K}_\alpha$ formed in $\mathbf{D}(\mathrm{R})$. (You may assume that the diagram is indexed by the natural numbers, i.e. by the category $\{0 \to 1 \to 2 \cdots\}$, as opposed to a general filtered category.)

4. *10 points.* Using the universal characterization of the derived tensor product, compute $\mathbf{Z}/m\mathbf{Z} \otimes^{\mathrm{L}}_{\mathbf{Z}} \mathbf{Z}/n\mathbf{Z} \in \mathbf{D}(\mathbf{Z})$ in two cases: (1) $\gcd(m, n) = 1$; (2) $m = n$. (Hint: Note that $\mathbf{Z}/n\mathbf{Z}$ is isomorphic in $\mathbf{D}(\mathbf{Z})$ to the cofibre of the multiplication by n map $n : \mathbf{Z} \to \mathbf{Z}$.)

5. *10 points.* Let $\mathcal{C}$ be an R-linear cocomplete ∞-category. Given two objects C and D of $\mathcal{C}$, there exists a *mapping complex* $\underline{\mathrm{Maps}}_{\mathcal{C}}(\mathrm{C}, \mathrm{D}) \in \mathbf{D}(\mathrm{R})$ refining the mapping type $\mathrm{Maps}_{\mathcal{C}}(\mathrm{C}, \mathrm{D}) \in \mathbf{H}$. Specifically, $\underline{\mathrm{Maps}}_{\mathcal{C}}(-, -) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{D}(\mathrm{R})$ is uniquely characterized by the fact that it sends colimits in the second argument to colimits and colimits in the first variable to limits, and satisfies $\mathrm{Maps}_{\mathcal{C}}(\mathrm{K} \otimes \mathrm{C}, \mathrm{D}) \simeq \mathrm{Maps}_{\mathbf{D}(\mathrm{R})}(\mathrm{K}, \underline{\mathrm{Maps}}_{\mathcal{C}}(\mathrm{C}, \mathrm{D}))$ functorially in $\mathrm{K} \in \mathbf{D}(\mathrm{R})$.[1]

   Use this characterization to show that (1) the underlying type of $\underline{\mathrm{Maps}}_{\mathcal{C}}(\mathrm{C}, \mathrm{D})$ is indeed $\mathrm{Maps}_{\mathcal{C}}(\mathrm{C}, \mathrm{D})$; (2) for the R-linear ∞-category $\mathcal{C} = \mathbf{D}(\mathrm{R})$, the mapping complex is the same as the internal Hom in $\mathbf{D}(\mathrm{R})$.

   Generally, we will simply write $\mathrm{Maps}_{\mathcal{C}}(\mathrm{C}, \mathrm{D})$ for the mapping complex; this will not lead to much confusion.

6. *15 points.* Let X be a topological space and $\mathrm{K} \in \mathbf{D}(\mathrm{R})$ a complex. Consider the constant sheaf $\underline{\mathrm{K}}$, by definition the localization $\mathrm{L}(\mathrm{K}_{\mathrm{cst}})$ of the constant presheaf.
   (1) Show that $\mathrm{K}_{\mathrm{cst}}$ need not be a sheaf, i.e., that the unit map $\mathrm{K}_{\mathrm{cst}} \to \underline{\mathrm{K}}$ is not an isomorphism.
   (2) Let X be the infinite disjoint union $\coprod_n \mathrm{pt}$, indexed by natural numbers $n \geqslant 0$. Describe the sheaf $\underline{\mathrm{K}}$ in this case.
   (3) For a general topological space X, show that $\underline{\mathrm{K}}$ is given by $\Gamma(\mathrm{U}, \underline{\mathrm{K}}) \simeq \prod_{\pi_0(\mathrm{U})} \mathrm{K}$, the product over the connected components of U, for every open $\mathrm{U} \subset \mathrm{X}$.

[1] The expression $\mathrm{K} \otimes \mathrm{C}$ uses the action of $\mathbf{D}(\mathrm{R})$ on $\mathcal{C}$.