Exercise sheet 3

The minimum passing average is 20 points per sheet.

1. *20 points.* Let $\mathrm{K} \in \mathbf{D}(\mathrm{R})$ be a complex. Given an integer $k \geqslant 0$, consider the diagram

$$\mathrm{K} \xrightarrow{d} \mathrm{K}^{\oplus k} \rightrightarrows \mathrm{K}^{\oplus k^2} \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \cdots \tag{0.1}$$

as in the lecture. Recall that the morphism d is the diagonal $a \mapsto (a, \ldots, a)$.

(1) Define precisely the cosimplicial structure in (0.1). For any morphism $\alpha : [m] \to [n]$ in the simplex category $\mathbf{\Delta}$, describe the induced map

$$\alpha^* : \mathrm{K}^{\oplus k^{m+1}} \longrightarrow \mathrm{K}^{\oplus k^{n+1}}.$$

In particular, verify that the two coface maps

$$d^0, d^1 : \mathrm{K}^{\oplus k} \rightrightarrows \mathrm{K}^{\oplus k^2}$$

are given by

$$d^0(a_1, \ldots, a_k) = \begin{bmatrix} a_1 & a_1 & \cdots & a_1 \\ a_2 & a_2 & \cdots & a_2 \\ \vdots & \vdots & & \vdots \\ a_k & a_k & \cdots & a_k \end{bmatrix}, \qquad d^1(a_1, \ldots, a_k) = \begin{bmatrix} a_1 & a_2 & \cdots & a_k \\ a_1 & a_2 & \cdots & a_k \\ \vdots & \vdots & & \vdots \\ a_1 & a_2 & \cdots & a_k \end{bmatrix}.$$

(2) Let $\mathrm{K} : \mathrm{pt} \to \mathbf{D}(\mathrm{R})$ denote the constant functor sending the unique object of the trivial category pt to K. Define $\widetilde{\mathrm{K}} : \mathbf{\Delta} \to \mathbf{D}(\mathrm{R})$ to be the right Kan extension of K along the functor $\mathrm{pt} \to \mathbf{\Delta}$ that picks out the object $[0] \in \mathbf{\Delta}$. Show that $\widetilde{\mathrm{K}}$ coincides with the cosimplicial diagram $\mathrm{K}^{\oplus k^{\bullet+1}}$, by using the pointwise formula for right Kan extensions.

(3) Deduce that the totalization of $\mathrm{K}^{\oplus \bullet+1}$, i.e., the limit of $\tilde{\mathrm{K}}$, is the same as the limit of the constant functor $\mathrm{K} : \mathrm{pt} \to \mathbf{D}(\mathrm{R})$. In other words, (0.1) is a limit diagram.

2. *10 points.* In the situation of Exercise 1, let $\mathrm{K}_\blacksquare \in \mathrm{Ch}(\mathrm{R})$ be a termwise projective chain complex representing K. Form the bicomplex obtained by applying the normalized chain complex functor of Dold–Kan to the cosimplicial object $(\mathrm{K}_\blacksquare)^{\oplus k^{\bullet+1}}$ in $\mathrm{Ch}(\mathrm{R})$. Show that the total complex of this bicomplex represents the totalization of (0.1). (This gives a different proof that (0.1) is a limit diagram.)

3. *10 points.* Let X and Y be topological spaces. Show that the functor

$$\mathrm{PShv}(\mathrm{X} \sqcup \mathrm{Y}) \to \mathrm{PShv}(\mathrm{X}) \times \mathrm{PShv}(\mathrm{Y}),$$

given informally by $\mathcal{F} \mapsto (\mathcal{F}|_\mathrm{X}, \mathcal{F}|_\mathrm{Y})$, is an equivalence. Show that it restricts to an analogous equivalence on Shv. (Hint: begin by describing the category $\mathrm{Op}(\mathrm{X} \sqcup \mathrm{Y})$.)

4. *20 points.* Let X be a topological space and $\mathcal{F} \in \mathrm{Shv}(\mathrm{X})$ a sheaf. The *stalk* of $\mathcal{F}$ at a point $x \in \mathrm{X}$ is the complex

$$\mathcal{F}_x := \varinjlim_{\mathrm{U} \ni x} \Gamma(\mathrm{U}, \mathcal{F}) \in \mathbf{D}(\mathrm{R}),$$

where the colimit is taken over all open neighbourhoods U of x. The *germ* of a section $s \in \Gamma(\mathrm{U}, \mathcal{F})$ is its image $s_x \in \mathcal{F}_x$.[1] The *support* of $\mathcal{F}$ is the subset $\mathrm{Supp}(\mathcal{F}) \subset \mathrm{X}$ containing all points $x \in \mathrm{X}$ such that $\mathcal{F}_x \not\simeq 0$ in $\mathbf{D}(\mathrm{R})$. The *support* of a section $s \in \Gamma(\mathrm{U}, \mathcal{F})$ is the subset $\mathrm{Supp}(s) \subset \mathrm{X}$ containing all points $x \in \mathrm{X}$ such that $s_x \not\simeq 0$.

(1) Show that $\mathrm{Supp}(\mathcal{F})$ need not be a closed subset of X, but that $\mathrm{Supp}(s)$ is always closed.[2]

[1] If K is a complex, the notation $s \in \mathrm{K}$ means that s is a point in the underlying type of K, i.e., a morphism $\mathrm{R} \to \mathrm{K}$ in $\mathbf{D}(\mathrm{R})$.

[2] 2025-11-05: The statement has been corrected.

(2) Let x be a point and $i_x : \{x\} \hookrightarrow \mathrm{X}$ the inclusion. Show that there is a canonical isomorphism

$$\Gamma(\{x\}, i_x^*(\mathcal{F})) \simeq \mathcal{F}_x$$

in $\mathbf{D}(\mathrm{R})$.

(3) Given a complex $\mathrm{K} \in \mathrm{D}(\mathrm{R})$, recall that the skyscraper $\underline{\mathrm{K}}_x$ at a point $x \in \mathrm{X}$ can be identified with $i_{x,*}(\mathrm{K})$. Compute the stalks $(\underline{\mathrm{K}}_x)_y$ for $y \neq x$ and for $y = x$, and describe $\mathrm{Supp}(\underline{\mathrm{K}}_x)$. (Warning: if X is not T_1, you will see that the support may not be just the point $\{x\}$.)

5. *20 points.* Let X be a topological space.

(1) Show that for every $\mathcal{F} \in \mathrm{Shv}(\mathrm{X})$, there is a canonical map

$$\Gamma_c(\mathrm{X}, \mathcal{F}) \to \Gamma(\mathrm{X}, \mathcal{F})$$

which is invertible if X is compact.

(2) Show that the canonical map $\Gamma_c(\mathrm{U} \sqcup \mathrm{V}, \mathcal{F}) \to \Gamma_c(\mathrm{U}, \mathcal{F}|_{\mathrm{U}}) \oplus \Gamma_c(\mathrm{V}, \mathcal{F}|_{\mathrm{V}})$ is invertible if $\mathrm{X} = \mathrm{U} \sqcup \mathrm{V}$.

(3) Assume X is Hausdorff. For any $\mathrm{K} \in \mathbf{D}(\mathrm{R})$ and any point $x \in \mathrm{X}$, show that the compactly supported global sections of the skyscraper sheaf are given by $\Gamma_c(\mathrm{X}, \underline{\mathrm{K}}_x) \simeq \mathrm{K}$.

(4) For any $\mathrm{K} \in \mathbf{D}(\mathrm{R})$ and any open $\mathrm{U} \subset \mathrm{X}$, show that $\Gamma_c(\mathrm{X}, \mathrm{cst}_{\mathrm{X}}^{\mathrm{U}}(\mathrm{K}))$ is isomorphic to K if U is relatively compact in X, and 0 otherwise.

6. *10 points.* Let $f : \mathrm{X} \to \mathrm{Y}$ be a morphism of topological spaces. For any finite set $[n] \in \mathbf{\Delta}$ and any collection of sheaves $\{\mathcal{G}_i\}_{i \in [n]}$ on Y, construct a canonical isomorphism

$$f^*\big(\bigotimes_i \mathcal{G}_i\big) \simeq \bigotimes_i f^*(\mathcal{G}_i).$$

7. *15 points.*

(1) Let X be a discrete topological space. For every $\mathrm{K} \in \mathbf{D}(\mathrm{R})$, show that there are canonical isomorphisms

$$\Gamma(\underline{\mathrm{K}}_{\mathrm{X}}) \simeq \prod_{x \in \mathrm{X}} \mathrm{K},$$

$$\Gamma_c(\underline{\mathrm{K}}_{\mathrm{X}}) \simeq \bigoplus_{x \in \mathrm{X}} \mathrm{K}.$$

In particular, the forget supports map $\Gamma_c(\underline{\mathrm{K}}_{\mathrm{X}}) \to \Gamma(\underline{\mathrm{K}}_{\mathrm{X}})$ is invertible if and only if either X is finite, or $\mathrm{K} \simeq 0$.

(2) More generally, let $f : \mathrm{X} \to \mathrm{Y}$ be a morphism of discrete spaces. Given a sheaf $\mathcal{F} \in \mathrm{Shv}(\mathrm{X})$, describe the $*$- and !-pushforwards

$$f_*(\mathcal{F}), f_!(\mathcal{F}) \in \mathrm{Shv}(\mathrm{Y}).$$

8. *20 points.* Let $\mathrm{X} = (0, 1)$ with the usual topology.

(1) Show that any compact closed subset $\mathrm{A} \subset (0, 1)$ is contained in some smaller open interval (a, b) which is relatively compact.

(2) For every $\mathrm{K} \in \mathbf{D}(\mathrm{R})$, show that $\mathrm{R}\Gamma_c(\mathrm{X}, \underline{\mathrm{K}}) \simeq \mathrm{K}$.