Exercise sheet 4

The minimum passing average is 20 points per sheet.

1. *5 points.* Let R be a commutative ring, and consider the forgetful functor $\mathbf{D}(R) \to \mathbf{H}$, $K \mapsto K^\circ$, sending a complex to its underlying homotopy type. Recall that this functor is limit-preserving.
   (1) Show that $K \mapsto K^\circ$ is not conservative. That is, given a morphism $\phi : K \to L$ in $\mathbf{D}(R)$, the invertibility of $\phi^\circ : K^\circ \to L^\circ$ need not imply invertibility of ϕ.
   (2) Show that the functors
   $$\mathbf{D}(R) \to \mathbf{H}, \quad K \mapsto (K[-i])^\circ$$
   are jointly conservative as i ranges over nonnegative integers.
   (3) Let X be a topological space and $\mathcal{F} \in \mathrm{PShv}(X; R)$. Show that $\mathcal{F}$ is a sheaf if and only if the $\mathbf{H}$-valued presheaves
   $$U \mapsto (\Gamma(U, \mathcal{F})[-i])^\circ$$
   are sheaves for all $i \geqslant 0$.

2. *10 points.* Let $X = [0, 1]$ be the closed unit interval and $i : \{0, 1\} \hookrightarrow X$ the inclusion of the boundary. Show that $i^!(\underline{R}_X) \simeq 0$.

3. *15 points.* Let X be a locally compact Hausdorff space. Show that if $X = Y \cup Z$ where Y and Z are closed subsets, then there is a long exact sequence
   $$\cdots \xrightarrow{\partial} \mathrm{H}^{\mathrm{BM}}_*(Y \cap Z) \to \mathrm{H}^{\mathrm{BM}}_*(Y) \oplus \mathrm{H}^{\mathrm{BM}}_*(Z) \to \mathrm{H}^{\mathrm{BM}}_*(X) \xrightarrow{\partial} \mathrm{H}^{\mathrm{BM}}_{*-1}(Y \cap Z) \to \cdots$$

4. *5 points.* Recall that $\mathrm{Perf}(R) \subseteq \mathbf{D}(R)$ is the smallest full subcategory which contains R, and is closed under finite colimits, shifts, and direct summands. Show that a complex $K \in \mathbf{D}(R)$ belongs to $\mathrm{Perf}(R)$ if and only if it can be represented by a chain complex $K_\bullet \in \mathrm{Ch}(R)$ which is bounded and has all terms K_n finitely generated projective as R-modules.

5. *10 points.*
   (1) Show that the tensor product on $\mathbf{D}(R)$ restricts to $\mathrm{Perf}(R)$: if $K, L \in \mathbf{D}(R)$ are both perfect, then the tensor product $K \otimes L \in \mathbf{D}(R)$ is also perfect.
   (2) Assume that R contains no nontrivial idempotents. Given a perfect complex $K \in \mathrm{Perf}(R)$, show that the following two conditions are equivalent: (a) K is *invertible*, i.e., there exists an object $L \in \mathrm{Perf}(R)$ such that $K \otimes L \simeq R$; and (b) K is isomorphic to $M[n]$ where $M \in \mathrm{Mod}_R$ is an invertible R-module (i.e., a finitely generated projective R-module of rank one) and $n \in \mathbf{Z}$.

6. *5 points.* Let X be a topological space. Show that if X is a disjoint union of n points, then $\mathrm{Loc}^\diamond(X)$ is equivalent to a product of n copies of $\mathbf{D}(R)$. Show that even if X is connected, then $\Gamma(X, -) : \mathrm{Loc}^\diamond(X) \to \mathbf{D}(R)$ need not be an equivalence.

7. *10 points.* The *fundamental groupoid* $\Pi_1(X)$ of a topological space X is the category whose objects are points $x \in X$ and morphisms $x \to y$ are equivalence classes of paths $\gamma : [0, 1] \to X$ with $\gamma(0) = x$, $\gamma(1) = y$, modulo endpoint-preserving homotopy. The composition law is defined by concatenation of paths.
   Consider the punctured complex plane $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$. Compute $\Pi_1(\mathbf{C}^*)$ as follows:
   (1) Show that $\mathbf{C}^*$ deformation-retracts onto the unit circle $\{z \in \mathbf{C} : |z| = 1\}$. In particular, all points of $\mathbf{C}^*$ are path-connected.
   (2) Show that $\pi_1(\mathbf{C}^*, 1)$ is cyclic, generated by the loop
   $$\gamma(t) = e^{2\pi i t}, \qquad t \in [0, 1].$$

Conclude that $\Pi_1(\mathbf{C}^*)$ is equivalent to the groupoid $B\mathbf{Z}$, which has a single object $*$, $\mathrm{Hom}(*, *) = \mathbf{Z}$, and the composition law is addition.

(3) Describe the data encoded by a functor $\Pi_1(\mathrm{C}^*) \to \mathbf{D}(R)$.