Exercise sheet 5

The minimum passing average is 20 points per sheet.

1. *5 points.*
   (1) Show that if X is a connected topological space and $U \subseteq X$ is a connected open, then the restriction map
   $$\mathrm{res} : \Gamma(X, \underline{K}_X) \to \Gamma(U, \underline{K}_X)$$
   induces an isomorphism $\mathrm{H}^0(X, \underline{K}_X) \simeq \mathrm{H}^0(U, \underline{K}_X)$ for every complex $K \in \mathbf{D}(R)$.
   (2) Show that this need not hold if U is not connected.

2. *5 points.* Let R be a commutative ring and $x \in R$ an element. Let $K \in \mathbf{D}(R)$ be the complex represented by the chain complex $K_\bullet \in \mathrm{Ch}(R)$ with $K_0 = R$, $K_1 = R$, differential $d_1 : K_1 \to K_0$ given by the multiplication by x map $x : R \to R$, and all other terms zero.[1] Describe the underlying type $K^\circ \in \mathbf{H}$ in terms of the underlying type R° (which is discrete, i.e., a set). (Hint: describe K as a co/fibre of a morphism in $\mathbf{D}(R)$.)

3. *15 points.* Let $\mathbf{C}^* = \mathbf{C}\backslash\{0\}$ be the punctured complex plane.
   (1) Let $U_+ \subseteq \mathbf{C}^*$ and $U_- \subseteq \mathbf{C}^*$ be the complements of $\{z \in \mathbf{C}^* \mid \mathrm{Re}(z) \leqslant 0\}$ and $\{z \in \mathbf{C}^* \mid \mathrm{Re}(z) \geqslant 0\}$, respectively. Use Čech descent for the open cover $\mathbf{C}^* = U_+ \cup U_-$, and the equivalence $\mathrm{Loc}^\diamond(X) \simeq \mathrm{Loc}^\diamond(\mathrm{pt})$ for a contractible space X, to compute $\mathrm{C}^\bullet(\mathbf{C}^*; R) = \Gamma(\mathbf{C}^*; \underline{R}_{\mathbf{C}^*})$.
   (2) Let $\mathcal{F} \in \mathrm{Loc}^\diamond(\mathbf{C}^*)$ be a locally constant sheaf on $\mathbf{C}^*$. Consider its monodromy representation, which encodes a complex $K \in \mathbf{D}(R)$ (the stalk at some point x) together with an automorphism $T : K \xrightarrow{\sim} K$ (the parallel transport with respect to a loop generating $\pi_1(\mathbf{C}^*, x)$). Compute the complex of global sections $\Gamma(\mathbf{C}^*, \mathcal{F})$, using descent as in part (i) but taking the potentially nontrivial monodromy T into account.

[1] This is the *Koszul complex* $\mathrm{Kosz}_R(x)$.