Exercise sheet 6

The minimum passing average is 20 points per sheet. This sheet is due by the morning of **Tuesday 12/9**.

1. *10 points.* Let $X = [0,1]$ be the unit interval.
   (1) Let $\mathcal{F} \in \mathrm{Shv}(X)$ be a locally constant sheaf which is discrete, i.e., $\mathcal{F} \in \mathrm{Loc}^{\diamond}(X)_{[0,0]}$. Show that there exists a finite open cover $X = \bigcup_{1 \leqslant i \leqslant n} U_i$ such that each $\mathcal{F}|_{U_i}$ is constant, satisfying moreover the following properties:
      (a) Each U_i is an open interval.
      (b) $0 \in U_1$, and for $i > 1$ each $U_i \cap U_{i-1}$ is a nonempty open interval.
      (c) Each union $V_i := U_1 \cup U_2 \cup \cdots \cup U_i$ is an open interval.
   (2) Show that for each i, the commutative square of restriction maps

$$\begin{array}{ccc} \Gamma(V_i, \mathcal{F}) & \longrightarrow & \Gamma(V_{i-1}, \mathcal{F}) \\ \downarrow & & \downarrow \\ \Gamma(U_i, \mathcal{F}) & \longrightarrow & \Gamma(U_i \cap V_{i-1}, \mathcal{F}) \end{array}$$

   is a pullback square in $\mathbf{D}(R)$, and all arrows are isomorphisms.
   (3) Deduce that $\mathcal{F}$ is constant on X.

2. *10 points.* Let $S^1 = \{z \in \mathbf{C} \mid |z| = 1\}$ and let $f : S^1 \to S^1$ be the map $z \mapsto z^n$. Compute the monodromy of the sheaf $f_*(\underline{R})$.

3. *10 points.* Suppose given a pullback square of topological spaces

$$\begin{array}{ccc} X' & \xrightarrow{g} & Y' \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{f} & Y. \end{array}$$

   Prove the following special cases of the base change isomorphism $\mathrm{Ex}^*_! : g_! p^* \simeq q^* f_!$.
   (1) If $f = j$ and $g = j'$ are open embeddings, show that there is a canonical isomorphism $\mathrm{Ex}^*_! : j'_! p^* \to q^* j_!$.
   (2) If $f = i$ and $g = i'$ are closed embeddings, so that $i_* \simeq i_!$, $i'_* \simeq i'_!$, show using the localization triangle that there is a canonical isomorphism $\mathrm{Ex}^*_* : q^* i_* \to i'_* p^*$.