Exercise sheet 8

The minimum passing average is 20 points per sheet.

1. *5 points.* Derive the following re-interpretation of Poincaré duality for !-pullback. For any smooth scheme X over $\mathbf{C}$ of dimension d, there exists a *fundamental class* $[X] \in \mathrm{H}^{\mathrm{BM}}_{2d}(X;R)$ such that
$$(-) \cap [X] : \mathrm{H}^k(X;R) \to \mathrm{H}^{\mathrm{BM}}_{k-2d}(X;R)$$
is an isomorphism. Here, $\cap$ denotes the *cap product*, which is an action of cohomology on Borel–Moore homology (in particular, explain how to define this action in terms of the six functor formalism).

2. *5 points.* Let X be a smooth scheme over $\mathbf{C}$ and let $Z \subseteq X$ be a smooth closed subscheme of codimension c. Denote by $i : Z \hookrightarrow X$ the inclusion. Show that there is a long exact sequence
$$\cdots \xrightarrow{\partial} \mathrm{H}^{k-2c}(Z) \xrightarrow{i_*} \mathrm{H}^k(X) \to \mathrm{H}^k(U) \xrightarrow{\partial} \mathrm{H}^{k-2c+1}(Z) \xrightarrow{i_*} \cdots$$
called the *Gysin sequence.* The morphism $i_* : \mathrm{H}^{*-2c}(Z) \to \mathrm{H}^*(X)$ is called the *Gysin map* in cohomology.

3. *10 points.*
   (1) Let $i : \mathrm{pt} \hookrightarrow \mathbf{A}^1$ be the inclusion of the origin, where $\mathrm{pt} = \mathrm{Spec}(\mathbf{C})$, and let $j : \mathbf{G}_m \hookrightarrow \mathbf{A}^1$ be the inclusion of the complement. Compute $i^* j_*(\underline{R})$.
   (2) Generalize the calculation to the case of a vector bundle $E \to X$, with $i : X \hookrightarrow E$ the zero section and $j : \mathring{E} \hookrightarrow E$ the inclusion of the complement.
   (3) Let $i : \mathrm{pt} \hookrightarrow \mathbf{A}^1$ and $j : \mathbf{G}_m \hookrightarrow \mathbf{A}^1$ be as in (1). Let $\mathcal{L}$ be a *locally* constant sheaf on $\mathbf{G}_m$. Using the equivalence $\mathrm{Loc}^\diamond(\mathbf{C}^*;R) \simeq \mathrm{Fun}(\Pi_\infty(\mathbf{C}^*), \mathbf{D}(R))$, identify $\mathcal{L}$ with an object $K \in \mathbf{D}(R)$ together with a monodromy automorphism $T : K \xrightarrow{\sim} K$, or equivalently an object $\widetilde{K} \in \mathbf{D}(R[T,T^{-1}])$.
   Show that $i^* j_*(\mathcal{L}) \in \mathrm{Shv}(\mathrm{pt};R) \simeq \mathbf{D}(R)$ is identified with the *homotopy invariants* $\widetilde{K}^{hT}$. (Equivalently, this is the mapping complex $\underline{\mathrm{Maps}}_{R[T,T^{-1}]}(R,\widetilde{K})$, where R is regarded as an $R[T,T^{-1}]$-module via the trivial augmentation $T \mapsto 1$.)

4. *5 points.*
   (1) Show that !-pushforward satisfies the *Künneth formula*, i.e., for any two morphisms $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ in $\mathrm{Sch}^{\mathrm{lft}}_{\mathbf{C}}$, we have
$$f_{1,!}(\mathcal{F}_1) \boxtimes f_{2,!}(\mathcal{F}_2) \simeq (f_1 \times f_2)_!(\mathcal{F}_1 \boxtimes \mathcal{F}_2),$$
   where $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is the induced morphism[1].
   (2) Calculate $\Gamma_c(\mathbf{A}^n; \underline{R})$, using Künneth to reduce to the case of $\mathbf{A}^1$, and then to the topological space $\mathbf{R}$, which is homeomorphic to $(0,1)$.

[1] $X \times Y$ denotes the fibred product over $\mathrm{Spec}(\mathbf{C})$.