Exercise sheet 9

The minimum passing average is 20 points per sheet.

1. *5 points. Formation of stalks is t-exact.* That is, let X be a topological space and let $x \in X$ be a point. Write also $x : \mathrm{pt} \hookrightarrow X$ for the inclusion. Then the functor $x^* : \mathrm{Shv}(X; R) \to \mathrm{Shv}(\mathrm{pt}; R) \simeq \mathbf{D}(R)$ is t-exact.

   Hint: if $\mathcal{F}$ is n-coconnective, check that the mapping type $\mathrm{Maps}(R[n+1], x^*\mathcal{F})$ is contractible by expressing $x^*\mathcal{F}$ as a certain colimit over open neighbourhoods $U \ni x$, and using the fact that $R[n+1]$ is a *compact* object of $\mathbf{D}(R)$.

2. *5 points. Connectivity can be detected on stalks.* That is, let X be a topological space and $\mathcal{F} \in \mathrm{Shv}(X)$ a sheaf. Then $\mathcal{F}$ is n-connective if and only if $x^*\mathcal{F}$ is n-connective for every point $x \in X$.

   Hint: note that $\mathcal{F}$ is n-connective if and only if $\tau_{\leqslant n-1}(\mathcal{F}) \simeq 0$. Note that $\tau_{\leqslant n-1}(\mathcal{F})$ lies in the left completion $\widehat{\mathrm{Shv}}(X)$, and recall that a morphism $\mathcal{F} \to \mathcal{G}$ in $\widehat{\mathrm{Shv}}(X)$ is invertible if and only if $x^*(\mathcal{F}) \to x^*(\mathcal{G})$ is invertible for every point $x \in X$.

3. *5 points.* Give an example of a proper morphism $f : X \to Y$ in $\mathrm{Sch}^{\mathrm{lft}}_{\mathbf{C}}$ such that f_* is not t-exact.

4. *5 points.* Check that f^*, for any morphism $f : X \to Y$ in $\mathrm{Sch}^{\mathrm{lft}}_{\mathbf{C}}$, preserves constructibility. Check that if $\mathcal{F}, \mathcal{G} \in \mathrm{Shv}_{\mathrm{c}}(X)$, then also $\mathcal{F} \otimes \mathcal{G}$ is constructible.

5. *10 points.* Let $X \in \mathrm{Sch}^{\mathrm{lft}}_{\mathbf{C}}$ and let $X = \bigcup_\alpha U_\alpha$ be an open cover. Check that $\mathcal{F} \in \mathrm{Shv}(X)$ is constructible iff $\mathcal{F}|_{U_\alpha}$ is constructible for all α.

6. *5 points.* Show that the previous exercise fails for analytic open covers.