SELECTED SOLUTIONS

Sheet 6, no. 1. *Let $X = [0,1]$ be the unit interval.*

(1) *Let $\mathcal{F} \in \mathrm{Shv}(X)$ be a locally constant sheaf which is discrete, i.e., $\mathcal{F} \in \mathrm{Loc}^{\diamond}(X)_{[0,0]}$. Show that there exists a finite open cover $X = \bigcup_{1 \leqslant i \leqslant n} U_i$ such that each $\mathcal{F}|_{U_i}$ is constant, satisfying moreover the following properties:*
   (a) *Each U_i is an open interval.*
   (b) *$0 \in U_1$, and for $i > 1$ each $U_i \cap U_{i-1}$ is a nonempty open interval.*
   (c) *Each union $V_i := U_1 \cup U_2 \cup \cdots \cup U_i$ is an open interval.*

(2) *Show that for each i, the commutative square of restriction maps*

$$\begin{array}{ccc} \Gamma(V_i, \mathcal{F}) & \longrightarrow & \Gamma(V_{i-1}, \mathcal{F}) \\ \downarrow & & \downarrow \\ \Gamma(U_i, \mathcal{F}) & \longrightarrow & \Gamma(U_i \cap V_{i-1}, \mathcal{F}) \end{array}$$

is a pullback square in $\mathbf{D}(R)$, and all arrows are isomorphisms.

(3) *Deduce that $\mathcal{F}$ is constant on X.*

Solution. We explain the proof of (2). In fact, we will let $\mathcal{F} \in \mathrm{Loc}^{\diamond}(X)$ be arbitrary, not necessarily discrete.

The square is a pullback by the sheaf condition for $\mathcal{F}$. Since $\mathcal{F}|_{U_i}$ is constant, so is $\mathcal{F}|_{U_i \cap V_{i-1}}$. Let $K \in \mathbf{D}(R)$ such that $\mathcal{F} \simeq \underline{K}_{U_i}$. Recall that since U_i is contractible, the functor $a_i^* : \mathrm{Shv}(\mathrm{pt}) \to \mathrm{Shv}(U_i)$ is fully faithful, where $a_i : U_i \to \mathrm{pt}$ is the projection. In particular, we have $\Gamma(U_i, \mathcal{F}) \simeq a_{i,*} a_i^*(K) \simeq K$. The same holds for the interval V_{i-1}, as well as the (nonempty) intersection of intervals $U_i \cap V_{i-1}$. It follows that the lower and right-hand restriction maps are identified with $\mathrm{id}_K : K \to K$. Since the square is a pullback, the upper and left-hand arrows are thus also isomorphisms.

Lemma 1. *Let X be a contractible topological space. Then $\mathcal{F}$ is constant if and only if the counit $a^* a_*(\mathcal{F}) \to \mathcal{F}$ is an isomorphism, where $a : X \to \mathrm{pt}$ is the projection.*

Proof. The condition is clearly sufficient, as it means $\mathcal{F} \simeq \underline{K}_X$ where $K := a_*(\mathcal{F}) \simeq \Gamma(X, \mathcal{F})$.

Conversely, suppose $\mathcal{F}$ is constant, i.e., $\mathcal{F} \simeq \underline{K}_X \simeq a^*(K)$ for some $K \in \mathbf{D}(R)$. Consider the diagram

$$a^*(K) \xrightarrow{a^*\mathrm{unit}} a^* a_* a^*(K) \xrightarrow{\mathrm{counit}(a^*K)} a^*(K).$$

By the triangle identities for the adjunction (a^*, a_*), this composite is the identity of $a^*(K)$. Since X is contractible, the unit $\mathrm{id} \to a_* a^*$ is an isomorphism. By the

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two-of-three property for isomorphisms, it follows that the second arrow is also an isomorphism. □

Lemma 2. *Let X be a topological space and $\mathcal{F} \in \mathrm{Shv}(X)$. Let $X = \bigcup_i U_i$ be an open cover such that each U_i is contractible, each $\mathcal{F}|_{U_i}$ is constant, and each restriction map*

$$\Gamma(X, \mathcal{F}) \to \Gamma(U_i, \mathcal{F})$$

is an isomorphism. Then the counit $a^*a_*(\mathcal{F}) \to \mathcal{F}$ is an isomorphism. In particular, $\mathcal{F}$ is constant.

Proof. Let $j_i : U_i \hookrightarrow X$ be the inclusions and $a : X \to \mathrm{pt}$, $a_i : U_i \to \mathrm{pt}$ the projections. By assumption, the units $a_*(\mathcal{F}) \to a_*j_{i,*}j_i^*(\mathcal{F}) \simeq a_{i,*}(\mathcal{F}|_{U_i})$ are isomorphisms. To show that $a^*a_*(\mathcal{F}) \to \mathcal{F}$ is an isomorphism it will suffice to show that it restricts to an isomorphism over every open U_i. Each such restriction is identified with

$$a_i^*a_*(\mathcal{F}) \to \mathcal{F}|_{U_i}.$$

Using the isomorphism $a_*(\mathcal{F}) \simeq a_{i,*}(\mathcal{F}|_{U_i})$, we may rewrite this as

$$\mathrm{counit} : a_i^*a_{i,*}(\mathcal{F}|_{U_i}) \to \mathcal{F}|_{U_i},$$

which is invertible in view of the previous lemma, since $\mathcal{F}|_{U_i}$ is constant and U_i is contractible. □

Finally, to prove (3), let us show by induction that $\mathcal{F}|_{V_i}$ is constant for each i. The case $i = 1$ holds by assumption. Assume $i > 1$ and $\mathcal{F}|_{V_{i-1}}$ is constant. Now U_i and V_{i-1} are both contractible (they are open intervals), so by (2) we may apply the previous lemma to deduce that $\mathcal{F}|_{V_i}$ is constant. In particular, $\mathcal{F}$ is constant on $V_n = X$.

Sheet 7, no. 1. *Show that $a^* : \mathrm{Loc}^\diamond(\mathrm{pt}) \to \mathrm{Loc}^\diamond([0,1])$ is an equivalence, where $a : [0,1] \to \mathrm{pt}$ is the projection, by reducing to the case of discrete locally constant sheaves considered in Sheet 6. (Note that the claim is equivalent to the assertion that every locally constant sheaf on $[0,1]$ is constant.)*

Solution. Recall that for any space X, the functor a^* is right t-exact, where $a : X \to \mathrm{pt}$. We begin by observing that it is in fact t-exact (even though X need not be a topological manifold, i.e., a need not be a topological submersion):

Lemma 3. *Let X be a topological space and consider the projection $a : X \to \mathrm{pt}$. Then the functor $a^* : \mathrm{Shv}(\mathrm{pt}) \to \mathrm{Shv}(X)$ is t-exact.*

Proof. It will suffice to show that a^* is left t-exact, i.e., preserves coconnectivity. Assume first that $K \in \mathbf{D}(R)_{[-n,0]} \simeq \mathrm{Shv}(\mathrm{pt})_{[-n,0]}$ for some $n \geqslant 0$. We argue by induction on n that $a^*(K) \simeq \underline{K}_X$ lies in $\mathrm{Shv}(X)_{[-n,0]}$. For $n = 0$ the claim follows from the fact that $\underline{M}_X \in \mathrm{Shv}(X)_{[0,0]}$ for any $M \in \mathrm{Mod}_R$ (see Prop. 3.6.45 in the lecture notes). For $n > 0$, consider the exact triangle

$$\tau_{\geqslant -n+1}(K) \to K \to \tau_{-n}(K) \simeq \mathrm{H}_{-n}(K)[-n].$$

By the induction hypothesis, a^* sends the left-hand term to $\mathrm{Shv}(X)_{[-n+1,0]}$. By the $n = 0$ case (and shifting), it sends the right-hand term to $\mathrm{Shv}(X)_{[-n,-n]}$. It follows that $a^*(K)$ is an extension of objects in $\mathrm{Shv}(X)_{[-n,0]}$ and hence itself belongs to $\mathrm{Shv}(X)_{[-n,0]}$.

Now suppose that $K \in \mathbf{D}(R)_{\leqslant 0}$. We may write $K \simeq \varinjlim_{n \geqslant 0} \tau_{\geqslant -n}(K)$. Since a^* is a left adjoint, it preserves colimits and so $a^*(K) \simeq \varinjlim_{n \geqslant 0} a^*(\tau_{\geqslant -n}(K))$. By the discussion above, each term lies in $\mathrm{Shv}(X)_{[-n,0]}$ and in particular is coconnective. Since this is a *filtered* colimit, and the standard t-structure is compatible with filtered colimits (by Prop. 3.6.37), we deduce that $a^*(K)$ is coconnective. □

Recall that, since the space $I = [0,1]$ is a CW complex, a sheaf $\mathcal{F} \in \mathrm{Shv}(I)$ is locally constant if and only if the cohomology sheaves $\mathcal{H}^i(\mathcal{F})$ are discrete locally constant sheaves for all $i \in \mathbf{Z}$. In particular, $\mathcal{F}$ is locally constant if and only if $\tau_{\geqslant -n}(\mathcal{F})$ is locally constant for all $n \geqslant 0$. Thus, the limit diagram (witnessing the right completeness of the standard t-structure)

$$\mathrm{Shv}(I) \to \cdots \to \mathrm{Shv}(I)_{\geqslant -2} \xrightarrow{\tau_{\geqslant -1}} \mathrm{Shv}(I)_{\geqslant -1} \xrightarrow{\tau_{\geqslant 0}} \mathrm{Shv}(I)_{\geqslant 0}$$

restricts to $\mathrm{Loc}^{\diamond}(I) \simeq \varprojlim_{n \geqslant 0} \mathrm{Loc}^{\diamond}(I)_{\geqslant -n}$. The same is obviously true for pt in place of I, so we have the following diagram where both rows are limit diagrams:

$$\begin{array}{ccccccccc}
\mathrm{Loc}^{\diamond}(\mathrm{pt}) & \longrightarrow \cdots \longrightarrow & \mathrm{Loc}^{\diamond}(\mathrm{pt})_{\geqslant -2} & \xrightarrow{\tau_{\geqslant -1}} & \mathrm{Loc}^{\diamond}(\mathrm{pt})_{\geqslant -1} & \xrightarrow{\tau_{\geqslant 0}} & \mathrm{Loc}^{\diamond}(\mathrm{pt})_{\geqslant 0} \\
\downarrow a^* & & \downarrow a^* & & \downarrow a^* & & \downarrow a^* \\
\mathrm{Loc}^{\diamond}(I) & \longrightarrow \cdots \longrightarrow & \mathrm{Loc}^{\diamond}(I)_{\geqslant -2} & \xrightarrow{\tau_{\geqslant -1}} & \mathrm{Loc}^{\diamond}(I)_{\geqslant -1} & \xrightarrow{\tau_{\geqslant 0}} & \mathrm{Loc}^{\diamond}(I)_{\geqslant 0}
\end{array}$$

Since a^* is right t-exact, it restricts to functors

$$a^* : \mathrm{Loc}^{\diamond}(\mathrm{pt})_{\geqslant -n} \to \mathrm{Loc}^{\diamond}(I)_{\geqslant -n} \tag{0.1}$$

for each n such that the diagram commutes. Thus, it will suffice to show that (0.1) is an equivalence for any n.

Similarly, since $\mathrm{Shv}(X)$ is also left complete where X is I or pt (since these are CW complexes), and a^* is also left t-exact, we also have the following commutative diagram where both rows are limit diagrams:

$$\begin{array}{ccccccccc}
\mathrm{Loc}^{\diamond}(\mathrm{pt}) & \longrightarrow \cdots \longrightarrow & \mathrm{Loc}^{\diamond}(\mathrm{pt})_{\leqslant 2} & \xrightarrow{\tau_{\leqslant 1}} & \mathrm{Loc}^{\diamond}(\mathrm{pt})_{\leqslant 1} & \xrightarrow{\tau_{\leqslant 0}} & \mathrm{Loc}^{\diamond}(\mathrm{pt})_{\leqslant 0} \\
\downarrow a^* & & \downarrow a^* & & \downarrow a^* & & \downarrow a^* \\
\mathrm{Loc}^{\diamond}(I) & \longrightarrow \cdots \longrightarrow & \mathrm{Loc}^{\diamond}(I)_{\leqslant 2} & \xrightarrow{\tau_{\leqslant 1}} & \mathrm{Loc}^{\diamond}(I)_{\leqslant 1} & \xrightarrow{\tau_{\leqslant 0}} & \mathrm{Loc}^{\diamond}(I)_{\leqslant 0}.
\end{array}$$

In particular, for any fixed $n \geqslant 0$, the functor (0.1) is the limit of $a^* : \mathrm{Loc}^{\diamond}(\mathrm{pt})_{[-n,m]} \to \mathrm{Loc}^{\diamond}(I)_{[-n,m]}$ over $m \geqslant 0$.

Thus let $\mathcal{F} \in \mathrm{Loc}^{\diamond}(I)_{[-n,m]}$ and let us show that $\mathcal{F}$ is constant. Shifting, we may as well assume that $\mathcal{F} \in \mathrm{Loc}^{\diamond}(I)_{[0,b]}$ for some integer $b \geqslant 0$. We argue by induction on b, the case $b = 0$ already known from the previous exercise. Let $b > 0$ and consider the exact triangle

$$\mathcal{H}^{-b}(\mathcal{F})[b] \simeq \tau_{\geqslant b}(\mathcal{F}) \to \mathcal{F} \to \tau_{\leqslant b-1}(\mathcal{F}).$$

Since the right-hand term lies in $\mathrm{Loc}^{\diamond}(I)_{[0,b-1]}$, the induction hypothesis yields that it is constant. Since the left-hand term lies in $\mathrm{Loc}^{\diamond}(I)_{[0,0]}$ up to shifting, the case $b = 0$ implies that it is also constant. So it will suffice to show that any extension of constant sheaves is constant.

Lemma 4. *Let X be a contractible topological space and suppose given an exact triangle*

$$\underline{K}_X \to \mathcal{F} \to \underline{L}_X$$

in $\mathrm{Shv}(X)$*, where* $K, L \in \mathbf{D}(R)$*. Then* $\mathcal{F}$ *is constant.*

Proof. Rotating, we may write $\mathcal{F}$ as the cofibre of the boundary map $\partial : \underline{L}_X[-1] \to \underline{K}_X$. Using the fully faithfulness of $a^* : \mathrm{Shv}(\mathrm{pt}) \to \mathrm{Shv}(X)$, we may write

$$\mathrm{Maps}_{\mathrm{Shv}(X)}(\underline{L}_X[-1], \underline{K}_X) \simeq \mathrm{Maps}_{\mathrm{Shv}(X)}(a^*(L[-1]), a^*(K)) \simeq \mathrm{Maps}_{\mathbf{D}(R)}(L[-1], K).$$

Thus the boundary map ∂ can be written as $a^*(\partial)$ for some $\partial : L[-1] \to K$ in $\mathbf{D}(R)$. Setting $M := \mathrm{Cofib}(\partial : L[-1] \to K)$, this latter can be regarded as the boundary map for the exact triangle

$$K \to M \to L.$$

By exactness of a^*, we thus have $\underline{M}_X = a^*(M) \simeq \mathcal{F}$. □

(In fact, what this shows is that the full subcategory of $\mathrm{Shv}(X)$ spanned by constant sheaves is a *stable* subcategory, when X is contractible.)

Sheet 8, no. 3.

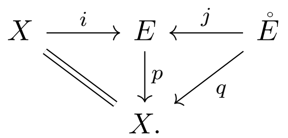
(1) *Let $i : \mathrm{pt} \hookrightarrow \mathbf{A}^1$ be the inclusion of the origin, where* $\mathrm{pt} = \mathrm{Spec}(\mathbf{C})$*, and let $j : \mathbf{G}_m \hookrightarrow \mathbf{A}^1$ be the inclusion of the complement. Compute $i^* j_*(\underline{R})$.*

(2) *Generalize the calculation to the case of a vector bundle $E \to X$, with $i : X \hookrightarrow E$ the zero section and $j : \mathring{E} \hookrightarrow E$ the inclusion of the complement.*

(3) *Let $i : \mathrm{pt} \hookrightarrow \mathbf{A}^1$ and $j : \mathbf{G}_m \hookrightarrow \mathbf{A}^1$ be as in (1). Let $\mathcal{L}$ be a* locally *constant sheaf on $\mathbf{G}_m$. Using the equivalence* $\mathrm{Loc}^{\diamond}(\mathbf{C}^*; R) \simeq \mathrm{Fun}(\Pi_\infty(\mathbf{C}^*), \mathbf{D}(R))$*, identify $\mathcal{L}$ with an object $K \in \mathbf{D}(R)$ together with a monodromy automorphism $T : K \xrightarrow{\sim} K$, or equivalently an object $\widetilde{K} \in \mathbf{D}(R[T, T^{-1}])$.*

Show that $i^* j_*(\mathcal{L}) \in \mathrm{Shv}(\mathrm{pt}; R) \simeq \mathbf{D}(R)$ is identified with the homotopy invariants *$\widetilde{K}^{hT}$. (Equivalently, this is the mapping complex $\underline{\mathrm{Maps}}_{R[T,T^{-1}]}(R, \widetilde{K})$, where R is regarded as an $R[T, T^{-1}]$-module via the trivial augmentation $T \mapsto 1$.)*

Solution. We explain the proof of (2). Let $p : E \to X$ be a vector bundle and consider the diagram



We begin by recalling the dual Euler transformation

$$i^! p^* := \Sigma^{-E} \xrightarrow{\mathrm{eul}_E^\vee} \mathrm{id},$$

which is dual to $\mathrm{eul}_E : \mathrm{id} \to \Sigma^E$. Recall that this is defined as

$$i^! p^* \xrightarrow{\mathrm{Ex}^{*!}} i^* p^* \simeq \mathrm{id}$$

using the exchange transformation $\mathrm{Ex}^{*!} : i^! \to i^*$ coming from the pullback square

$$\begin{array}{ccc} X & = & X \\ \| & & \downarrow{\scriptstyle i} \\ X & \xrightarrow{i} & E. \end{array}$$

Unravelling the definitions, this is given by the composite

$$\mathrm{Ex}^{*!} : \mathrm{id}^* i^! \xrightarrow{\text{unit}} \mathrm{id}_! \mathrm{id}_! \mathrm{id}^* i^! \simeq \mathrm{id}^! i^* i_! i^! \xrightarrow{\text{counit}} \mathrm{id}^! i^*,$$

where the isomorphism is the base change formula for the pullback square above. In other words, the dual Euler transformation can be expressed as

$$i^! p^* \simeq i^* i_* i^! p^* \xrightarrow{\text{counit}} i^* p^* \simeq \mathrm{id}.$$

In particular, it fits into the localization triangle

$$i^! p^* \xrightarrow{\mathrm{eul}_E^\vee} i^* p^* \xrightarrow{\text{unit}} i^* j_* j^* p^* \simeq i^* j_* q^*.$$

We conclude that:

Lemma 5. *Let $p : E \to X$ be a vector bundle over a scheme $X \in \mathrm{Sch}_{\mathbf{C}}^{\mathrm{lft}}$. Then with notation as above,*

$$i^* j_* q^*(-) \simeq \mathrm{Cofib}(\mathrm{eul}_E^\vee : \Sigma^{-E} \to \mathrm{id}).$$

In particular, $i^* j_*(\underline{R}) \simeq \mathrm{Cofib}(\mathrm{eul}_E^\vee : \underline{R}\langle -E \rangle \to \underline{R})$ in $\mathrm{Shv}(X)$.

Using the inverse Σ^E to Σ^{-E}, we have the identification of mapping complexes

$$\underline{\mathrm{Hom}}(\underline{R}\langle -E \rangle, \underline{R}) \simeq \underline{\mathrm{Hom}}(\underline{R}, \underline{R}\langle E \rangle) \simeq \mathrm{C}^\bullet(X; R)\langle E \rangle,$$

under which $\mathrm{eul}_E^\vee : \underline{R}\langle -E \rangle \to \underline{R}$ is identified with the Euler class $e(E)$. Via the Thom isomorphism $\underline{R}\langle E \rangle \simeq \underline{R}[2r]$, we may further identify

$$\mathrm{C}^\bullet(X; R)\langle E \rangle \simeq \mathrm{C}^\bullet(X; R)[2r],$$

where $r = \mathrm{rk}(E)$, under which $e(E)$ corresponds to the *top Chern class* $c_{2r}(E)$. Given a null-homotopy $e(E) \simeq 0$, we thus obtain

$$i^* j_*(\underline{R}) \simeq \mathrm{Cofib}(\underline{R}[-2r] \xrightarrow{0} \underline{R}) \simeq \underline{R}[-2r+1] \oplus \underline{R}.$$

For example, recall that there is a canonical such null-homotopy whenever $p : E \to X$ admits a nowhere zero section s. This is in particular the case when E admits $\mathbf{A}^1_X$ as a direct summand, so in particular we have $i^* j_*(\underline{R}) \simeq R \oplus R[-1]$ in the case of (1).

In general however, $i^* j_*(\underline{R})$ is just an extension via the exact triangle

$$\underline{R} \to i^* j_*(\underline{R}) \xrightarrow{c_{2r}(E)} \underline{R}[-2r+1]$$

which is non-split if $c_{2r}(E)$ is nonzero. For example, let $X = \mathbf{P}^1_{\mathbf{C}}$ and let E be the total space of $\mathcal{O}(2)$. Then $c_1(E) \in \mathrm{C}^\bullet(\mathbf{P}^1_{\mathbf{C}}; R)[2]$ is nonzero, as can be seen e.g. by inspection of the projective bundle formula.