CONSTRUCTIBLE SHEAVES AND VANISHING CYCLES

ADEEL A. KHAN

1. Introduction

In these lectures we will introduce the theories of constructible sheaves and vanishing cycles. They can be summed up as a package consisting of the following collection of structures:

$\mathrm{Shv}(X)$	sheaves	X an algebraic variety over $\mathbf{C}$
$\mathrm{Shv}_{\mathrm{c}}(X)$	constructible sheaves	
$(\otimes, \underline{\mathrm{Hom}})$	tensor and internal Hom	
$\mathbb{D}$	Verdier duality	
(f^*, f_*)	$*$-pullback and $*$-pushforward	$f : X \to Y$ a morphism
$(f_!, f^!)$	!-pushforward and !-pullback	
ψ_t, ϕ_t	nearby and vanishing cycles	$t : X \to \mathbf{A}^1$ a regular function

For every complex algebraic variety X, we will introduce a category $\mathrm{Shv}(X)$ as well as a full subcategory $\mathrm{Shv}_{\mathrm{c}}(X)$ of *constructible* objects. These will be equipped with several different operations. For each X, we will have the operations of tensor product and internal Hom, satisfying the adjunction formula

$$\mathrm{Hom}(\mathscr{F} \otimes \mathscr{G}, \mathscr{H}) \simeq \mathrm{Hom}(\mathscr{F}, \underline{\mathrm{Hom}}(\mathscr{G}, \mathscr{H})).$$

We will also have a duality functor $\mathbb{D}$, satisfying $\mathbb{D}(\mathbb{D}(\mathscr{F})) \simeq \mathscr{F}$ for $\mathscr{F}$ constructible. For a morphism of complex algebraic varieties $f : X \to Y$, we will have two different types of pullbacks and pushforwards, satisfying the adjunction formulas

$$\mathrm{Hom}(f^*(\mathscr{G}), \mathscr{F}) \simeq \mathrm{Hom}(\mathscr{G}, f_*(\mathscr{F}))$$
$$\mathrm{Hom}(f_!(\mathscr{F}), \mathscr{G}) \simeq \mathrm{Hom}(\mathscr{F}, f^!(\mathscr{G})).$$

Date: 2025-10-16.

Finally, for a regular function t on X, i.e., a morphism $t : X \to \mathbf{A}^1$ to the complex affine line, we will have the functors of nearby and vanishing cycles, ψ_t and ϕ_t respectively. All these various operations restrict to constructible objects, and are woven together through various compatibilities. For example, the $*$- and !-functors are interchanged by Verdier duality, and similarly

$$\mathbb{D}(\mathscr{F} \otimes \mathscr{G}) \simeq \underline{\mathrm{Hom}}(\mathscr{F}, \mathbb{D}(\mathscr{G})).$$

This package is often referred to informally as the *six functor formalism*; while the name refers to the six functors $\otimes, \underline{\mathrm{Hom}}, f^*, f_*, f_!, f^!$, the term is used loosely and is often understood to also include the closely related functors $\mathbb{D}$, ψ_t, and ϕ_t.

Our first goal in these lectures will be to learn how this formalism is constructed, and how to effectively manipulate and work with it. We will then illustrate how this abstract machinery forms an effective language for the *cohomological* study of the geometry and topology of complex algebraic varieties. For example, we will see how classical results such as the Gauss–Bonnet and Lefschetz trace formulas find elegant expression (and proofs) in this language. We will also see how the nearby and vanishing cycles functors are used to analyze the topology of degenerating families of algebraic varieties.

2. Sheaves on spaces

Let X be a topological space and R a commutative ring. Our goal in this section will be to develop the construction

$$\mathrm{Shv}(X; R),$$

whose objects are R-linear "sheaves" on X. Later on, we will apply this to the topological space underlying a complex algebraic variety[1], but for now our considerations will not make use of the algebraic structure on X.

For example, there will be a constant sheaf

$$\underline{R}_X \in \mathrm{Shv}(X; R),$$

which $*$-pushes forward along the projection $f : X \to \mathrm{pt}$ to a sheaf

$$f_*(\underline{R}_X) \in \mathrm{Shv}(\mathrm{pt}; R).$$

There will be an equivalence

$$\mathrm{Shv}(\mathrm{pt}; R) \xrightarrow{\sim} \mathbf{D}(R),$$

where objects of $\mathbf{D}(R)$ are represented by (co)chain complexes of R-modules. The object $f_*(\underline{R}_X)$ will correspond to an object

$$\mathrm{C}^\bullet(X; R) \in \mathbf{D}(R),$$

represented by the complex of singular cochains on X with coefficients in R. In fact, objects of $\mathbf{D}(R)$ are at first approximation "(co)chain complexes of R-modules *up to quasi-isomorphism*"; somewhat more precisely, $\mathbf{D}(R)$ is a certain enhancement of the derived category of R.

[1]more precisely, to the topological space $\mathsf{X}(\mathbf{C})$ of complex points of a finite type $\mathbf{C}$-scheme X

2.1. **Calculus of complexes.** Classical linear algebra provides a robust calculus for vector spaces, or more generally R-modules, through operations like kernels, cokernels, direct sums, and tensor products. A succinct way to summarize this is provided by the language of category theory: the collection of modules over a commutative ring R assembles into a *symmetric monoidal abelian category* Mod_R.

Derived linear algebra is a similar calculus for chain complexes of R-modules. While the collection of chain complexes of R-modules does similarly assemble into a symmetric monoidal abelian category $\mathrm{Ch}(R)$, it turns out that this "naïve" calculus is too rigid for our purposes. We need a *homotopical* version of this calculus, where equalities are replaced by chain homotopies.

To justify our claim that the naïve calculus is too rigid, let us consider the following example. Suppose X is a topological space and $\mathrm{C}_\bullet(X)$ is the complex of singular chains on X, regarded as an object of $\mathrm{Ch}(R)$. Given any continuous map $f : X \to Y$, there is an induced morphism of chain complexes $f_* : \mathrm{C}_\bullet(X) \to \mathrm{C}_\bullet(Y)$. This construction is functorial, so if we are given an open cover $X = U \cup V$, there is an induced commutative square

$$\begin{tikzcd} \mathrm{C}_\bullet(U\cap V) \arrow[r, "i^{U\cap V}_{V,*}"] \arrow[d, "i^{U\cap V}_{U,*}"'] & \mathrm{C}_\bullet(V) \arrow[d, "i^{V}_{X,*}"] \\ \mathrm{C}_\bullet(U) \arrow[r, "i^{U}_{X,*}"] & \mathrm{C}_\bullet(X) \end{tikzcd} \tag{2.1.1}$$

in $\mathrm{Ch}(R)$. However, we have:

Exercise 2.1.2. The square (2.1.1) is not a pushout in $\mathrm{Ch}(R)$. Equivalently, the sequence

$$0 \to \mathrm{C}_\bullet(U\cap V) \xrightarrow{\begin{bmatrix} i^{U\cap V}_{U,*} \\ i^{U\cap V}_{V,*} \end{bmatrix}} \mathrm{C}_\bullet(U) \oplus \mathrm{C}_\bullet(V) \xrightarrow{i^{U}_{X,*} - i^{V}_{X,*}} \mathrm{C}_\bullet(X) \to 0$$

is not exact in $\mathrm{Ch}(R)$.

Informally speaking, singular chains are not local. Nevertheless, we do have the following result (a reformulation of the Mayer–Vietoris theorem):

Theorem 2.1.3. *The square (2.1.1) is a* homotopy *pushout. Equivalently, the diagram*

$$\mathrm{C}_\bullet(U\cap V) \xrightarrow{\begin{bmatrix} i^{U\cap V}_{U,*} \\ i^{U\cap V}_{V,*} \end{bmatrix}} \mathrm{C}_\bullet(U) \oplus \mathrm{C}_\bullet(V) \xrightarrow{i^{U}_{X,*} - i^{V}_{X,*}} \mathrm{C}_\bullet(X)$$

determines an exact triangle.

As we will see, the notion of *exact triangle* is an "up to coherent homotopy" analogue of the notion of exact sequence. The meaning of this statement is that (2.1.1) is universal among all squares of the form

$$\begin{tikzcd} C_\bullet(U\cap V) \arrow[r, "i^{U\cap V}_{V,*}"] \arrow[d, "i^{U\cap V}_{U,*}"] & C_\bullet(V) \arrow[d, "\phi^V"'] \\ C_\bullet(U) \arrow[r, "\phi^U"] & K_\bullet \end{tikzcd}$$

that commute *up to coherent homotopy*: more precisely, among all tuples $(K_\bullet, \phi^U, \phi^V, H)$ where H is a specified chain homotopy $\phi^U \circ i^{U\cap V}_{U,*} \simeq \phi^V \circ i^{U\cap V}_{V,*}$, i.e., a degree one map $H : C_\bullet(U \cap V) \to K_{\bullet+1}$ satisfying

$$\phi^U \circ i^{U\cap V}_{U,*} - \phi^V \circ i^{U\cap V}_{V,*} = d \circ H + H \circ d.$$

We say that this homotopy H *witnesses* the commutativity of the above square.

The moral of this discussion is that for our purposes, the category $\mathrm{Ch}(R)$ should be replaced by some more elaborate homotopical structure, which encodes, instead of diagrams that commute up to *equality*, diagrams that commute up to *coherent homotopy*. This will have the effect that limits and colimits are replaced by homotopy limits and colimits. This is precisely what is achieved by the construction we have denoted by $\mathbf{D}(R)$, which assembles chain complexes of R-modules into a *symmetric monoidal stable ∞-category*. In particular, (2.1.1) will indeed be a pushout square in the ∞-category $\mathbf{D}(R)$.

2.1.1. *The ∞-category of complexes.* Before introducing the ∞-category $\mathbf{D}(R)$, we recall its classical approximation, the *derived category* of chain complexes of R-modules.

Reminder 2.1.4. Denote by $\mathrm{hD}(R)$ the localization of the category $\mathrm{Ch}(R)$ with respect to quasi-isomorphisms.[2] This is a category equipped with a functor $\gamma : \mathrm{Ch}(R) \to \mathrm{hD}(R)$ which sends all quasi-isomorphisms in $\mathrm{Ch}(R)$ to isomorphisms in $\mathrm{hD}(R)$, and is *initial* among all functors with this property. It admits various more explicit models, one of which is the following:

- Objects are K-injective complexes of R-modules.
- Morphisms $M_\bullet \to N_\bullet$ are morphisms in $\mathrm{Ch}(R)$, modulo the relation of chain homotopy.

Here, an object $M_\bullet \in \mathrm{Ch}(R)$ is called *K-injective* (following Spaltenstein) if for every $N_\bullet \in \mathrm{Ch}(R)$, every subcomplex $N'_\bullet \subseteq N_\bullet$ which is quasi-isomorphic to $N_\bullet$, and every morphism $f : N'_\bullet \to M_\bullet$, there exists a morphism $\overline{f} : N_\bullet \to M_\bullet$ extending f. This is an analogue for chain complexes of the notion of *injective* R-module.

[2] Classically, this category is denoted by $\mathbf{D}(R)$. In these notes this notation is instead used for the ∞-categorical derived category. Since the classical derived category can be realized as the *homotopy category* of the ∞-categorical one, we use the notation $\mathrm{hD}(R)$ for the former.

Our goal is to construct a factorization of γ through an intermediate structure,

$$\mathrm{Ch}(R) \to \mathbf{D}(R) \to \mathrm{hD}(R).$$

Informally speaking:

- $\mathrm{Ch}(R)$ is a structure encoding diagrams of chain complexes that commute up to equality.
- $\mathrm{hD}(R)$ is a structure encoding diagrams of chain complexes that commute up to homotopy.
- The intermediate construction $\mathbf{D}(R)$ should be a structure encoding diagrams of chain complexes that commute up to *coherent* homotopy.

In order to encode homotopy coherent diagrams of chain complexes, we must pass from ordinary categories to a richer kind of structure.

Construction 2.1.5. Consider the following collection of sets:

- Let S_0 denote the set of chain complexes of R-modules.
- Let S_1 denote the set of tuples $(M_\bullet, N_\bullet, \phi)$ where $M_\bullet$ and $N_\bullet$ are K-injective complexes and $\phi : M_\bullet \to N_\bullet$ is a morphism of chain complexes.
- Let S_2 denote the set of tuples $(M_\bullet, N_\bullet, P_\bullet, f_{01}, f_{12}, f_{02}, H)$ where $M_\bullet$, $N_\bullet$, and $P_\bullet$ are K-injective complexes, f_{01}, f_{12}, f_{02} are morphisms of chain complexes, and H is a chain homotopy $f_{02} \simeq f_{12} \circ f_{01}$.
- $\cdots$
- For any $n \geqslant 0$, we define S_n as follows. Informally, its elements are K-injective complexes $(M_0)_\bullet, (M_1)_\bullet, \dots, (M_n)_\bullet$, along with the morphisms $f_{ij} : (M_i)_\bullet \to (M_j)_\bullet$ which are compatible up to coherent homotopy. More precisely, the *coherence* means that we need to exhibit specific homotopies witnessing this compatibility. Thus, an element of S_n consists of K-injective complexes $(M_0)_\bullet, (M_1)_\bullet, \dots, (M_n)_\bullet$, together with a collection of maps[3]

$$f_I : (M_{i_-})_k \to (M_{i_+})_{k+m},$$

indexed by subsets $I = \{i_- < i_1 < \cdots < i_m < i_+\}$ of $\{0, 1, \dots, n\}$, satisfying the relation

$$d(f_I(x)) = (-1)^m f_I(dx) + \sum_{1 \leqslant j \leqslant m} (-1)^j \big(f_{I \smallsetminus \{i_j\}}(x) - (f_{i_j, \dots, i_+} \circ f_{i_-, \dots, i_j})(x) \big)$$

for every $x \in (M_{i_-})_k$.

There are also various natural maps between these S_n's.

- For every order-preserving map $\alpha : [m] \to [n]$, where $[n]$ is the set $\{0, 1, \dots, n\}$ with its usual order, we have a map $\alpha^* : S_n \to S_m$. Given $(M_j)_\bullet$, $0 \leqslant j \leqslant n$, and the maps $\{f_I\}_I$, α^* sends this to the

[3] not necessarily a morphism of chain complexes, i.e., not required to commute with the differentials

complexes $(M_{\alpha(j)})_\bullet$, $0 \leqslant j \leqslant n$, together with the collection of maps $\{g_J\}_J$ defined as follows:

$$g_J(x) := \begin{cases} f_{\alpha(J)} & \text{if } \alpha|_J \text{ is injective,} \\ x & \text{if } J = \{j, j'\} \text{ and } \alpha(j) = \alpha(j'), \\ 0 & \text{otherwise.} \end{cases}$$

This kind of structure has a name.

Definition 2.1.6. A *simplicial set* $S_\bullet$ is a collection of sets S_n, $n \geqslant 0$, and a collection of maps $\alpha^* : S_n \to S_m$ for every order-preserving map $\alpha : [m] \to [n]$. These maps α^* are required to be compatible with composition. The elements of S_n are called the *n-simplices* of $S_\bullet$.

Example 2.1.7. If $\mathscr{C}$ is a category, there is a simplicial set $N(\mathscr{C})_\bullet$ called the *nerve* of $\mathscr{C}$, whose n-simplices are chains of composable arrows

$$C_0 \to C_1 \to \cdots \to C_n$$

in $\mathscr{C}$. This simplicial set $N(\mathscr{C})_\bullet$ remembers everything there is to know about the category $\mathscr{C}$: its objects, its morphisms, and the composition law. More precisely, the construction $\mathscr{C} \mapsto N(\mathscr{C})_\bullet$ determines a fully faithful functor from the category of categories to the category of simplicial sets.

Construction 2.1.8. The homotopy theory of topological spaces yields a variant of Construction 2.1.5, where the 0-simplices are topological spaces, the 1-simplices are continuous maps, and chain homotopies are replaced by usual homotopies for the higher simplices.

These examples suggest that it is possible to regard simplicial sets as category-like structures, with objects given by the 0-simplices, morphisms given by the 1-simplices, and the higher simplices encoding higher coherent homotopies. Taking this approach leads to one rigourous definition of the notion of ∞-category, as developed in [Lur3] or [Ci].

In these notes, we will instead take a more informal approach that suffices for the "working mathematician". The simplicial sets of Constructions 2.1.5 and 2.1.8 should determine ∞-categories $\mathbf{D}(R)$ and $\mathbf{H}$, respectively.[4] Objects of $\mathbf{H}$ are homotopy types of topological spaces, and $\mathbf{H}$ is an ∞-categorical enhancement of the homotopy category of topological spaces in the same way that $\mathbf{D}(R)$ enhances the derived category of chain complexes.

Definition 2.1.9. An *∞-category* $\mathscr{C}$ is a collection of objects $X \in \mathscr{C}$, together with *mapping types*

$$\mathrm{Maps}_{\mathscr{C}}(X, Y) \in \mathbf{H}, \quad \text{for every } X, Y \in \mathscr{C},$$

and a composition law

$$\mathrm{Maps}_{\mathscr{C}}(X, Y) \times \mathrm{Maps}_{\mathscr{C}}(Y, Z) \to \mathrm{Maps}_{\mathscr{C}}(X, Z), \quad \text{for every } X, Y, Z \in \mathscr{C},$$

which is unital and associative *up to coherent homotopy.*

[4] In Construction 2.1.5, one could equivalently use *K-projective* rather than K-injective complexes. That is, the two resulting simplicial sets would define the same ∞-category, up to equivalence of ∞-categories.

Similarly, an *R-linear ∞-category* $\mathscr{C}$ is a collection of objects together with *mapping complexes* $\mathrm{Maps}_{\mathscr{C}}(X,Y) \in \mathbf{D}(R)$, and a composition law which is unital and associative up to coherent homotopy.

By convention, we will adopt the following terminology:

Definition 2.1.10. A *homotopy type*, or simply a *type* for short, is an object of $\mathbf{H}$.[5] A *complex* (of R-modules) is an object of $\mathbf{D}(R)$.

By definition, every complex $K \in \mathbf{D}(R)$ can be represented by a *chain complex* $K_\bullet \in \mathrm{Ch}(R)$, and similarly every type can be represented by some topological space.

Definition 2.1.11. Given a complex $K \in \mathbf{D}(R)$, the *underlying type* of K is the mapping type

$$K^\circ := \mathrm{Maps}_{\mathbf{D}(R)}(R,K) \in \mathbf{H}.$$

We will also sometimes simply write K for K° when there is no risk of confusion. A *point* of K is a point of K°, i.e., a morphism $R \to K$ in $\mathbf{D}(R)$.

Since mapping types are covariantly functorial in the target, the assignment $K \mapsto K^\circ$ determines a canonical functor of ∞-categories

$$\mathbf{D}(R) \to \mathbf{H}.$$

Remark 2.1.12. By analogy, note that if M is an R-module, then $\mathrm{Hom}_R(R,M)$ is its underlying set (with the R-module structure forgotten).

2.1.2. *Homotopy coherent diagrams and co/limits.* The ∞-category $\mathbf{D}(R)$, resp. $\mathbf{H}$, is designed to encode homotopy coherent diagrams, i.e., diagrams of chain complexes (resp. topological spaces) that commute up to coherent homotopy.

Definition 2.1.13. Let $\mathscr{I}$ be an ∞-category (e.g., an ordinary category regarded as an ∞-category). Let $\mathscr{C}$ be an ∞-category. An *$\mathscr{I}$-shaped diagram* in $\mathscr{C}$ is a functor of ∞-categories $F : \mathscr{I} \to \mathscr{C}$.

Example 2.1.14. Let $\mathscr{I} = \Delta^2$ be the category with objects 0, 1, and 2, and exactly three nontrivial morphisms $0 \to 1$, $1 \to 2$, and $0 \to 2$, with $[0 \to 2] = [1 \to 2] \circ [0 \to 1]$. A Δ^2-shaped diagram in $\mathbf{D}(R)$ amounts to the data of:

- three complexes M_0, M_1, M_2;
- three morphisms $f_{01} : M_0 \to M_1$, $f_{12} : M_1 \to M_2$, and $f_{02} : M_0 \to M_2$;
- a homotopy $f_{02} \simeq f_{12} \circ f_{01}$.

[5] An important theorem in ∞-category theory is that $\mathbf{H}$ is equivalent to the ∞-category of *∞-groupoids*, i.e., ∞-categories where all morphisms are invertible. We will therefore sometimes use the terminologies *types* and *∞-groupoids* interchangeably. Note that homotopy theorists use the terminology *spaces* (this doesn't suit our purposes, as we are interested in invariants like the ∞-category of sheaves on a space, which does not depend on just the homotopy type.) Also, some communities also use the terminology *anima* or *animae*.

Thus, when we say that the diagram

$$\begin{tikzcd} M_0 \arrow[r, "f_{01}"] \arrow[dr, "f_{02}"'] & M_1 \arrow[d, "f_{12}"] \\ & M_2 \end{tikzcd}$$

commutes up to coherent homotopy, we mean that it is specified by a functor $\Delta^2 \to \mathbf{D}(R)$.

Definition 2.1.15. Let $\mathscr{C}$ be an ∞-category. A *commutative triangle* in $\mathscr{C}$ is a Δ^2-shaped diagram.

Remark 2.1.16. We will take the assertion of Example 2.1.14 for granted, but it can be justified using the simplicial set model for $\mathbf{D}(R)$ as follows. For each n, the *standard n-simplex* is the simplicial set Δ^n whose k-simplices are order-preserving maps $[k] \to [n]$. The simplicial set Δ^2 is nothing but the nerve of the category denoted Δ^2 in Example 2.1.14. It is convenient to express this using the language of presheaves. Denote by $\mathbf{\Delta}$ the category whose objects are finite sets $[n] := \{0, 1, \ldots, n\}$ for $n \geqslant 0$, and whose morphisms $[m] \to [n]$ are order-preserving maps. Then a simplicial set is nothing but a presheaf on $\mathbf{\Delta}$, i.e., a functor from $\mathbf{\Delta}^{\mathrm{op}}$ to the category of sets, and $\Delta^n = \mathrm{Hom}_{\mathbf{\Delta}}(-, [n])$ is the presheaf represented by $[n]$. Now the Yoneda lemma says that morphisms of simplicial sets $\Delta^2 \to S_\bullet$ are in bijection with 2-simplices of $S_\bullet$. Applying this to the simplicial set of Construction 2.1.5, we see in particular that a Δ^2-shaped diagram in $\mathbf{D}(R)$, which is encoded by a morphism of simplicial sets $\Delta^2 \to S_\bullet$, is the same thing as an element of S_2.

We can similarly describe diagrams of more general shapes, because a general category $\mathscr{I}$ can be built out of the standard n-simplexes Δ^n.

Example 2.1.17. Let $\mathscr{I} = \square$ denote the category with four objects 00, 10, 01, and 11, with morphisms as depicted:

$$\begin{tikzcd} 00 \arrow[r] \arrow[d] \arrow[dr] & 10 \arrow[d] \\ 01 \arrow[r] & 11, \end{tikzcd}$$

such that $[10 \to 11] \circ [01 \to 10] = [00 \to 11] = [01 \to 11] \circ [00 \to 01]$. A $\square$-shaped diagram in $\mathbf{D}(R)$ amounts to the data of a diagram

$$\begin{tikzcd} M_{00} \arrow[r, "f_0"] \arrow[d, "g_0"] \arrow[dr, "h"] & M_{10} \arrow[d, "g_1"] \\ M_{01} \arrow[r, "f_1"] & M_{11} \end{tikzcd}$$

that commutes up to homotopies $g_1 \circ f_0 \simeq h \simeq f_1 \circ g_0$.

Definition 2.1.18. Let $\mathscr{C}$ be an ∞-category. A *commutative square* in $\mathscr{C}$ is a $\square$-shaped diagram.

Remark 2.1.19. Example 2.1.17 can be justified using the simplicial model for $\mathbf{D}(R)$ as follows. Let Δ^1 denote the category with two objects 0, 1, and a unique nontrivial morphism $0 \to 1$ (note that its nerve is the standard 1-simplex Δ^1 from Remark 2.1.16). The category $\square$ is equivalent to $\Delta^1 \times \Delta^1$, so in particular its nerve is the simplicial set $\Delta^1 \times \Delta^1$. In the category of simplicial sets, there is a pushout square

$$\begin{tikzcd} \Delta^1 \arrow[r] \arrow[d] & \Delta^2 \arrow[d] \\ \Delta^1 \arrow[r] & \Delta^1 \times \Delta^1 \end{tikzcd}$$

which one may visualize as the subdivision

$$\begin{tikzcd} 00 \arrow[r] \arrow[d] \arrow[dr] & 10 \arrow[d] \\ 01 \arrow[r] & 11 \end{tikzcd}$$

of $\square \simeq \Delta^1 \times \Delta^1$ into two Δ^2's glued along the diagonal Δ^1. Thus, by Remark 2.1.16, a $\square$-shaped diagram in $\mathbf{D}(R)$ can be understood as two homotopies identifying $g_1 \circ f_0$ and $f_1 \circ g_0$ separately with the diagonal arrow h.

Definition 2.1.20. Let $\mathscr{I}$ be the category depicted as follows:

$$\begin{tikzcd} & 10 \arrow[d] \\ 01 \arrow[r] & 11. \end{tikzcd}$$

Let $X : \mathscr{I} \to \mathscr{C}$ be an $\mathscr{I}$-shaped diagram in an ∞-category $\mathscr{C}$, depicted as:

$$\begin{tikzcd} & X_{10} \arrow[d] \\ X_{01} \arrow[r] & X_{11}. \end{tikzcd}$$

Given an object $Y \in \mathscr{C}$, consider the type

$$\mathrm{Maps}_{\mathrm{Fun}(\mathscr{I},\mathscr{C})}(Y_{\mathrm{cst}}, X) \in \mathbf{H},$$

where Y_{cst} is the constant diagram $Y \to Y \leftarrow Y$. The *pullback* of the diagram $X_{01} \to X_{11} \leftarrow X_{10}$ is an object $P \in \mathscr{C}$ equipped with a natural transformation $\alpha : P_{\mathrm{cst}} \to X$, such that

$$\mathrm{Maps}_{\mathscr{C}}(Y, P) \to \mathrm{Maps}_{\mathrm{Fun}(\mathscr{I},\mathscr{C})}(Y_{\mathrm{cst}}, X),$$

given by

$$(f : Y \to P) \mapsto (Y_{\mathrm{cst}} \xrightarrow{f_{\mathrm{cst}}} P_{\mathrm{cst}} \xrightarrow{\alpha} X),$$

is an isomorphism in $\mathbf{H}$ for all objects $Y \in \mathscr{C}$.

Unravelling, note that a point of $\mathrm{Maps}(Y_{\mathrm{cst}}, X)$ amounts to two maps $Y \to X_{01}$, $Y \to X_{10}$, and a commutative square

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & X_{10} \arrow[d] \\ X_{01} \arrow[r] & X_{11} \end{tikzcd} \tag{2.1.21}$$

in $\mathscr{C}$. The pullback is thus an object P fitting into such a commutative square, such that for any other such square (2.1.21), there exists a map $Y \to P$, unique up to a contractible space of choices, making the following diagram commute (up to coherent homotopy):

$$\begin{tikzcd} Y \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & P \arrow[r] \arrow[d] & X_{10} \arrow[d] \\ & X_{01} \arrow[r] & X_{11}. \end{tikzcd}$$

Notation 2.1.22. Given a commutative diagram $X_{01} \to X_{11} \leftarrow X_{10}$ in $\mathscr{C}$, we will denote by $X_{01} \times_{X_{11}} X_{10} \in \mathscr{C}$ the pullback (when it exists). The ∞-categorical Yoneda lemma implies that, if the pullback does exist, it is unique up to a contractible space of choices.

More generally, we define:

Definition 2.1.23. Let $X : \mathscr{I} \to \mathscr{C}$ be a diagram in an ∞-category $\mathscr{C}$. The *limit* of X is an object P equipped with a natural transformation $\alpha : P_{\mathrm{cst}} \to X$ such that the map

$$\mathrm{Maps}_{\mathscr{C}}(Y, P) \to \mathrm{Maps}_{\mathrm{Fun}(\mathscr{I},\mathscr{C})}(Y_{\mathrm{cst}}, X),$$

given by

$$(f : Y \to P) \mapsto (Y_{\mathrm{cst}} \xrightarrow{f_{\mathrm{cst}}} P_{\mathrm{cst}} \xrightarrow{\alpha} X),$$

is invertible in $\mathbf{H}$.

Notation 2.1.24. Given a diagram $X : \mathscr{I} \to \mathscr{C}$, written informally as $\{X_i\}_{i \in \mathscr{I}}$, the limit of X is denoted by

$$\varprojlim X \quad \text{or} \quad \varprojlim_{i \in \mathscr{I}} X_i.$$

When it exists, it is unique up to a contractible space of choices.

Remark 2.1.25. Note that limits in $\mathbf{D}(R)$ correspond to *homotopy* limits of chain complexes, as opposed to actual limits in $\mathrm{Ch}(R)$.

Definition 2.1.26. Dually, given a diagram $X : \mathscr{I} \to \mathscr{C}$, the *colimit* is defined dually and denoted by

$$\varinjlim X \quad \text{or} \quad \varinjlim_{i \in \mathscr{I}} X_i.$$

In other words, $\varinjlim X$ is the limit of $X^{\mathrm{op}} : \mathscr{I}^{\mathrm{op}} \to \mathscr{C}^{\mathrm{op}}$. For example, we have a notion of *pushouts* of diagrams $X_{01} \leftarrow X_{00} \to X_{10}$, dual to pullbacks.

2.1.3. *Co/fibres and exact triangles of complexes.* We now return to the ∞-category $\mathbf{D}(R)$ and aim to understand some examples of limits and colimits more concretely in this setting.

Example 2.1.27. Let $\phi : K \to L$ be a morphism of complexes in $\mathbf{D}(R)$. The *fibre* of ϕ, denoted $\mathrm{Fib}(\phi)$, is defined as the pullback

$$\begin{tikzcd} \mathrm{Fib}(\phi) \arrow[r] \arrow[d] & K \arrow[d, "\phi"] \\ 0 \arrow[r] & L. \end{tikzcd}$$

Let $Y \in \mathbf{D}(R)$ be a complex. A morphism $Y \to \mathrm{Fib}(\phi)$ is by definition a commutative square

$$\begin{tikzcd} Y \arrow[r, "\psi"] \arrow[d] & K \arrow[d, "\phi"] \\ 0 \arrow[r] & L \end{tikzcd}$$

in $\mathbf{D}(R)$, i.e., a morphism $\psi : Y \to K$ together with a null-homotopy $H : \phi\circ\psi \simeq 0$. If K, L, and Y are presented by termwise projective chain complexes $K_\bullet$, $L_\bullet$, and $Y_\bullet$, H can be presented by a graded map $H_\bullet : Y_\bullet \to L_{\bullet+1}$ satisfying $\phi\circ\psi = d \circ H + H \circ d$. Let us attempt to build a chain complex $F_\bullet$ presenting $\mathrm{Fib}(\phi)$. If a map $Y_\bullet \to F_\bullet$ is to encode ψ and H, it seems reasonable to set $F_n := K_n \oplus L_{n+1}$ for every $n \in \mathbf{Z}$. It's not difficult to write down differentials $d_F : F_n \to F_{n-1}$ with the property that $d^2 = 0$, ψ is a chain map, and H is a chain homotopy; for example, we may take

$$d_F(a,b) := (d_K(a), \phi(a) - d_L(b)).$$

Then it is not difficult to show that the chain complex $F_\bullet \in \mathrm{Ch}(R)$ indeed represents the fibre $\mathrm{Fib}(\phi) \in \mathbf{D}(R)$. Note moreover that $F_\bullet$ is precisely the shifted mapping cone $\mathrm{Cone}(\phi)[-1]$.[6]

Example 2.1.28. Dually, we define the *cofibre* of $\phi : K \to L$ as the complex $\mathrm{Cofib}(\phi) \in \mathbf{D}(R)$ fitting in the pushout square

$$\begin{tikzcd} K \arrow[r, "\phi"] \arrow[d] & L \arrow[d] \\ 0 \arrow[r] & \mathrm{Cofib}(\phi). \end{tikzcd}$$

By similar reasoning, one may represent $\mathrm{Cofib}(\phi)$ by the chain complex $Q_\bullet$ given by $Q_n := L_n \oplus K_{n-1}$ and differentials

$$d_Q(a,b) = (d_L(a) + \phi(b), -d_K(b)).$$

Note that $Q_\bullet$ is precisely the mapping cone $\mathrm{Cone}(\phi)$.[7]

Corollary 2.1.29. *For any morphism $\phi : K \to L$ in the ∞-category $\mathbf{D}(R)$, there is a canonical isomorphism* $\mathrm{Fib}(\phi) \simeq \mathrm{Cofib}(\phi)[-1]$.

[6] up to the usual choice of sign conventions in the definition of $\mathrm{Cone}(\phi)$, which agree up to quasi-isomorphism

[7] again, up to sign conventions

Definition 2.1.30. An *exact triangle* in $\mathbf{D}(R)$ is a diagram

$$K \xrightarrow{\phi} L \xrightarrow{\psi} M$$

together with a null-homotopy $\psi \circ \phi \simeq 0$, such that either of the following equivalent conditions hold:

(a) The induced map $\mathrm{Cofib}(\phi) \to M$ is an isomorphism in $\mathbf{D}(R)$.

(b) The induced map $K \to \mathrm{Fib}(\psi)$ is an isomorphism in $\mathbf{D}(R)$.

Example 2.1.31. For any complex $K \in \mathbf{D}(R)$, we have exact triangles

$$K[-1] \to 0 \to K, \quad K \to 0 \to K[1]$$

in $\mathbf{D}(R)$.

Proposition 2.1.32. *Given a diagram*

$$K_{01} \xrightarrow{\phi} K_{11} \xleftarrow{\psi} K_{10},$$

the pullback P is canonically isomorphic to

$$\mathrm{Fib}(\phi - \psi : K_{01} \oplus K_{10} \to K_{11}).$$

Exercise 2.1.33. State and prove a dual version of Proposition 2.1.32, computing pushouts as cofibres.

Proposition 2.1.34. *Given a commutative square*

$$\begin{tikzcd} K_{00} \arrow[r] \arrow[d] & K_{10} \arrow[d] \\ K_{01} \arrow[r] & K_{11} \end{tikzcd}$$

in $\mathbf{D}(R)$, it is a pullback square if and only if it is a pushout square.

Proof. Combine Corollary 2.1.29, Proposition 2.1.32, and Exercise 2.1.33. □

Definition 2.1.35. A commutative square in $\mathbf{D}(R)$ is *exact* if it is a pullback, or equivalently if it is a pushout.

References

[Ci] D.-C. Cisinski, *Higher categories and homotopical algebra.* Cambridge Studies in Advances Mathematics **180** (2019).

[Lur1] J. Lurie, *Higher Topos Theory.* Ann. Math. Stud. **170** (2009).

[Lur2] J. Lurie, *Higher Algebra*, version of 2017-09-18. Available at: https://www.math.ias.edu/~lurie/papers/HA.pdf.

[Lur3] J. Lurie, *Kerodon*, https://kerodon.net.

Institute of Mathematics, Academia Sinica, Taipei 10617, Taiwan