Exercise sheet 1
Prove all the propositions and/or theorems below.

Adjoint Functors.

Proposition 1. *Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be an adjunction. Assume that the right adjoint G is fully faithful. If ever $\mathcal{C}$ is complete (cocomplete), so is $\mathcal{D}$.*

Given a functor between small categories $u : A \longrightarrow B$, we will write the functor of composition with u

$$u^* : \widehat{B} \longrightarrow \widehat{A}\,.$$

It has a left adjoint

$$u_! : \widehat{A} \longrightarrow \widehat{B}$$

obtained as the extension by colimits of the composition of u with the Yoneda embedding of B. In other words, the functor u is the unique functor which commutes with small colimits such that $u_!(h_a) = h_{u(a)}$ for any object a of A. One checks that the functor u^* also preserves small colimits, and thus has a right adjoint

$$u_* : \widehat{A} \longrightarrow \widehat{B}$$

Proposition 2. *The following three conditions are equivalent.*

(i) *The functor u is fully faithful.*
(ii) *The functor $u_!$ is fully faithful.*
(iii) *The functor u_* is fully faithful.*

Skeletons. We may apply the preceding construction in the following case. For any integer $n \geq 0$, let $\mathbf{\Delta}_{\leq n}$ be the full subcategory whose objects are the sets of the form $[k]$ for $k \leq n$, and let $i_n : \mathbf{\Delta}_{\leq n} \longrightarrow \mathbf{\Delta}$ be the inclusion functor. We thus have a sequence of adjoints $(i_n)_!$, i_n^* and $(i_n)_*$, and thus a pair of adjoint. One defines a pair of adjoint functors by setting, for any simplicial set:

$$sk_n(X) = (i_n)_!(i_n^*(X)) \quad \text{and} \quad cosk_n(X) = (i_n)_!(i_n^*(X))\,.$$

Note that, *a priori*, the simplicial set $sk_n(X)$ has to be distinguished from the skeleton $Sk_n(X)$ (which is defined as a subobject of X).

Proposition 3. *If, for a given integer $n \geq 0$, a morphism of simplicial sets $X \longrightarrow Y$ induces injections $X_k \longrightarrow Y_k$ for all $k \leq n$, and if $Sk_n(X) = X$, then it is a monomorphism.*

The preceding proposition is useful to prove:

Proposition 4. *The co-unit morphism $sk_n(X) \longrightarrow X$ induces a canonical isomorphism*

$$sk_n(X) \simeq Sk_n(X)\,.$$

Groupoids. Let *Gpd* denote the full subcategory of *Cat* spanned by small groupoids, and let $\iota : \mathit{Gpd} \hookrightarrow \mathit{Cat}$ denote the inclusion.

Proposition 5. *The functor ι admits a left adjoint π_1 and a right adjoint k.*

The right adjoint sends any category C to $k(C)$, its maximal subgroupoid. This is defined as the subcategory of C with the same objects and only the invertible morphisms.

The groupoid $\pi_1(C)$, sometimes called the *fundamental groupoid* of C, can be described as the localization of C at the set of all its morphisms (in the sense of Gabriel–Zisman).

Now consider the inclusion $i : \mathit{Set} \hookrightarrow \mathit{Gpd}$, where we view a set E as a groupoid $i(E)$ with $Ob(i(E)) = E$ and only identity morphisms.

Lemma 6. *The functor $i : \mathit{Set} \hookrightarrow \mathit{Gpd}$ admits a left adjoint π_0.*

The set $\pi_0(K)$ is called the set of connected components of K.

Proposition 7. *For a small category C, the following conditions are equivalent.*

(i) *The category C is a groupoid (i.e. all morphisms of C are invertible).*
(ii) *The simplicial set $N(C)$ is an ∞-groupoid.*
(iii) *The simplicial set $N(C)$ is a Kan complex.*
(iv) *There exists a presheaf X on the category of non-empty finite sets (with all maps as morphisms) whose restriction to $\mathbf{\Delta}$ is isomorphic to C.*