Exercise sheet 2

1. Homotopy cocartesian squares

1.1. Let $\mathbf{C}$ be a model category.

Let $[1]$ denote the free category generated by the oriented graph $0 \to 1$. Note that a diagram $[1] \to \mathbf{C}$ is nothing else than a morphism in $\mathbf{C}$. By [1, Prop. 8.11], the projective model structure on the category $\mathbf{C}^{[1]}$ exists. The proof shows that cofibrations can be described as follows.

Given two diagrams $c_\bullet$ and $d_\bullet$ on $[1]$, a morphism $f : c_\bullet \to d_\bullet$ corresponds to a commutative square

$$\begin{array}{ccc} c_0 & \longrightarrow & c_1 \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} \\ d_0 & \longrightarrow & d_1. \end{array}$$

The morphism f is a cofibration if and only if f_0 is a cofibration and the canonical morphism

$$d_0 \sqcup_{c_0} c_1 \to d_1$$

is a cofibration.

In particular, an object $c_\bullet$ of $\mathbf{C}^{[1]}$ is cofibrant if and only if c_0 is cofibrant in $\mathbf{C}$, and the morphism $c_0 \to c_1$ is a cofibration in $\mathbf{C}$.

1.2. Let $\ulcorner$ denote the free category generated by the oriented graph

$$\begin{array}{ccc} (0,0) & \longrightarrow & (1,0) \\ \downarrow & & \\ (0,1). & & \end{array}$$

Definition 1.2.1. *A* correspondence *in* $\mathbf{C}$ *is a diagram* $\ulcorner \to \mathbf{C}$.

In other words, a correspondence is a diagram of the form

$$\begin{array}{ccc} c_{0,0} & \xrightarrow{f} & c_{1,0} \\ \downarrow{\scriptstyle g} & & \\ c_{0,1} & & \end{array} \tag{1.1}$$

Exercise 1.2.2.

(i) *The projective model structure exists on the category* $\mathbf{C}^{\ulcorner}$ *of correspondences in* $\mathbf{C}$.

(ii) *A correspondence of the form (1.1) is cofibrant if and only if each object* $c_{i,j}$ *is cofibrant in* $\mathbf{C}$ *and each morphism* f *and* g *is cofibrant in* $\mathbf{C}$.

Hence we get a well-defined homotopy colimit functor

$$\mathrm{L}\varinjlim_{\ulcorner} : \mathbf{C}^{\ulcorner} \to \mathbf{C}$$

(see [1, 8.14]).

1.3. Let $\square$ denote the category $[1] \times [1]$, i.e. the free category generated by the oriented graph

$$\begin{array}{ccc} (0,0) & \longrightarrow & (1,0) \\ \downarrow & & \downarrow \\ (0,1) & \longrightarrow & (1,1). \end{array}$$

Note that a diagram on the category $\square$, i.e. a functor $\square \to \mathbf{C}$, is nothing else than a commutative square in $\mathbf{C}$.

Suppose we have a commutative square

$$\begin{array}{ccc} c_{0,0} & \longrightarrow & c_{1,0} \\ \downarrow & & \downarrow \\ c_{0,1} & \longrightarrow & d \end{array} \tag{1.2}$$

in $\mathbf{C}$. This can be viewed as a co-cone under $c_\bullet$, i.e. a morphism $c_\bullet \to d_\ulcorner$ of functors $\ulcorner \to \mathbf{C}$, where $c_\bullet$ denotes the restriction of the diagram along the inclusion $\ulcorner \to \square$, and $d_\ulcorner$ denotes the constant diagram on $\ulcorner$ valued in d. By adjunction the morphism $c_\bullet \to d_\ulcorner$ corresponds to a canonical morphism

$$\mathrm{L}\varinjlim_{\ulcorner}(c_\bullet) \to d \tag{1.3}$$

in the homotopy category $ho(\mathbf{C})$.

Definition 1.3.1. *We say that the commutative square (1.2) is* homotopy cocartesian *if the canonical morphism (1.3) is an isomorphism in $ho(\mathbf{C})$.*

According to [1, Example 8.21] this is equivalent to the statement that the morphism

$$\varinjlim_{\ulcorner}(\mathrm{Q}(c_\bullet)) \to d$$

is a weak equivalence in $\mathbf{C}$, where $\mathrm{Q}(c_\bullet)$ is a cofibrant replacement of $c_\bullet$ in $\mathbf{C}^\ulcorner$.

1.4. Note that the category $\mathbf{C}^\square$ is equivalent to $(\mathbf{C}^{[1]})^{[1]}$. Applying the discussion in Paragraph 1.1 twice, one sees:

Exercise 1.4.1.

(i) *The projective model structure on $\mathbf{C}^\square$ exists.*

(ii) *An object of $\mathbf{C}^\square = (\mathbf{C}^{[1]})^{[1]}$, a morphism $f : c_\bullet \to d_\bullet$ of morphisms in $\mathbf{C}$, corresponding to a commutative square*

$$\begin{array}{ccc} c_0 & \longrightarrow & c_1 \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} \\ d_0 & \longrightarrow & d_1, \end{array}$$

is cofibrant if and only if each object is cofibrant in $\mathbf{C}$, and each morphism is a cofibration in $\mathbf{C}$.

Note that it follows that any cocartesian square, where all the objects are cofibrant and all the morphisms are cofibrations, is homotopy cocartesian.

1.5. The inclusion $i : \ulcorner \hookrightarrow \square$ induces a Quillen adjunction

$$i_! : \mathbf{C}^{\ulcorner} \rightleftarrows \mathbf{C}^{\square} : i^*$$

where the right adjoint is restriction and the left adjoint is defined by extending by colimits.

Note that the restriction functor i^* preserves cofibrant objects.

Exercise 1.5.1. *A commutative square* K *is homotopy cocartesian if and only if the canonical morphism*

$$\mathrm{L}i_!(i^*\mathrm{K}) \to \mathrm{K}$$

is invertible in $ho(\mathbf{C}^{\square})$*.*

The condition of a square to be homotopy cocartesian can also be checked by taking a cofibrant replacement in $\mathbf{C}^{\square}$ of the whole diagram (instead of its restriction to $\mathbf{C}^{\ulcorner}$).

Exercise 1.5.2. *A commutative square* K *is homotopy cocartesian if and only if there is a cofibrant replacement* $\mathrm{Q}(\mathrm{K})$ *in* $\mathbf{C}^{\square}$ *which is cocartesian.*

1.6. The above can be used to demonstrate:

Exercise 1.6.1. *A commutative square*

$$\begin{array}{ccc} c_{0,0} & \longrightarrow & c_{1,0} \\ \downarrow & & \downarrow \\ c_{0,1} & \longrightarrow & c_{1,1} \end{array}$$

is homotopy cocartesian if and only if the commutative square

$$\begin{array}{ccc} c_{0,0} & \longrightarrow & c_{0,1} \\ \downarrow & & \downarrow \\ c_{1,0} & \longrightarrow & c_{1,1} \end{array}$$

is homotopy cocartesian.

Exercise 1.6.2. *Let*

$$\begin{array}{ccc} c_{0,0} & \longrightarrow & c_{1,0} \\ \downarrow & & \downarrow \\ c_{0,1} & \longrightarrow & c_{1,1} \end{array}$$

be a commutative square. If both horizontal morphisms are weak equivalences, then the square is homotopy cocartesian.

Exercise 1.6.3. *Let*

$$\begin{array}{ccccc} c_{0,0} & \longrightarrow & c_{1,0} & \longrightarrow & c_{2,0} \\ \downarrow & & \downarrow & & \downarrow \\ c_{0,1} & \longrightarrow & c_{1,1} & \longrightarrow & c_{2,1} \end{array}$$

be a diagram of commutative squares. Suppose that the left-hand square is homotopy cocartesian. Then the right-hand side is homotopy cocartesian if and only if the composite square is homotopy cocartesian.

2. The canonical model structure on the category of categories

2.1. Let Cat denote the category of small categories. Recall from [1, Prop. 4.11] that the nerve functor $\mathrm{N} : \mathrm{Cat} \hookrightarrow \mathrm{Set}_{\mathbf{\Delta}}$ is fully faithful and admits a left adjoint τ. It follows by an exercise from Sheet 1 that the category Cat admits colimits and limits (since $\mathrm{Set}_{\mathbf{\Delta}}$ does, as a presheaf category).

It turns out that Cat admits a unique model structure where the weak equivalences are equivalences of categories. This model structure can be described as follows:

- The weak equivalences are equivalences of categories $u : \mathbf{C} \xrightarrow{\sim} \mathbf{D}$.
- The cofibrations are functors $u : \mathbf{C} \to \mathbf{D}$ which induce injections $\mathrm{O}b(\mathbf{C}) \hookrightarrow \mathrm{O}b(\mathbf{D})$ on sets of objects.
- The fibrations are functors[1] $u : \mathbf{C} \to \mathbf{D}$ such that for any object $c \in \mathbf{C}$ and any isomorphism $f : u(c) \xrightarrow{\sim} d$, with d an object of $\mathbf{D}$, there exists an isomorphism $\tilde{f} : c \xrightarrow{\sim} c'$ in $\mathbf{C}$ such that $u(\tilde{f})$ coincides with f (up to equality).

We will take these as definitions of the classes of cofibrations and fibrations, and show that they form a well-defined model structure.

Note that every category will be cofibrant and fibrant in this model structure.

2.2. Let [0] be the free category generated by the graph 0 (i.e. the terminal category), and J the free category generated by the graph

$$0 \underset{g}{\overset{f}{\rightleftarrows}} 1$$

under the relations $fg = \mathrm{id}$ and $gf = \mathrm{id}$. Note that this is the fundamental groupoid $\pi_1([1])$ (see Sheet 1).

Exercise 2.2.1. *A functor $u : \mathbf{C} \to \mathbf{D}$ is a fibration if and only if it satisfies the right lifting property with respect to the canonical inclusion*

$$[0] \hookrightarrow \mathrm{J}$$

which sends 0 to 0.

A trivial cofibration (resp. trivial fibration) is a cofibration (resp. fibration) that is also a weak equivalence, i.e. an equivalence of categories.

Exercise 2.2.2.

(i) *A functor $u : \mathbf{C} \to \mathbf{D}$ is a trivial cofibration if and only if it is isomorphic to the inclusion of a full subcategory which is equivalent to $\mathbf{D}$.*

(ii) *A functor $u : \mathbf{C} \to \mathbf{D}$ is a trivial fibration if and only if it is fully faithful and induces a surjection $\mathrm{Ob}(\mathbf{C}) \to \mathrm{Ob}(\mathbf{D})$ on objects.*

2.3. The following exercises will verify the axioms of model structure:

Exercise 2.3.1.

(i) *The class of equivalences of categories is stable under the 2-of-3 property.*

(ii) *The class of equivalences of categories (resp. cofibrations, fibrations) is stable under retracts.*

(iii) *The class of cofibrations (resp. fibrations) has the left lifting property (resp. right lifting property) with respect to trivial fibrations (resp. trivial cofibrations).*

[1] These are sometimes called *isofibrations*.

Let $u : \mathbf{C} \to \mathbf{D}$ be a functor. Let $\mathrm{Q}(u)$ denote teh category whose set of objects is $\mathrm{O}b(\mathrm{Q}(u)) = \mathrm{O}b(\mathbf{C}) \sqcup \mathrm{O}b(\mathbf{D})$, and morphisms are defined by

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Q}(u)}(c, c') &= \mathrm{Hom}_{\mathbf{D}}(u(c), u(c')) && (c, c' \in \mathrm{O}b(\mathbf{C})), \\ \mathrm{Hom}_{\mathrm{Q}(u)}(d, d') &= \mathrm{Hom}_{\mathbf{D}}(d, d') && (d, d' \in \mathrm{O}b(\mathbf{D})), \\ \mathrm{Hom}_{\mathrm{Q}(u)}(c, d) &= \mathrm{Hom}_{\mathbf{D}}(u(c), d) && (c \in \mathrm{O}b(\mathbf{C}), d \in \mathrm{O}b(\mathbf{D})), \\ \mathrm{Hom}_{\mathrm{Q}(u)}(d, c) &= \mathrm{Hom}_{\mathbf{D}}(d, u(c)) && (c \in \mathrm{O}b(\mathbf{C}), d \in \mathrm{O}b(\mathbf{D})). \end{aligned}$$

Exercise 2.3.2. *Any functor $u : \mathbf{C} \to \mathbf{D}$ factors as a composite*

$$\mathbf{C} \xrightarrow{v} \mathrm{Q}(u) \xrightarrow{w} \mathbf{D},$$

where v is a cofibration and w is a trivial fibration.

Let $\mathrm{R}(u)$ denote the category whose objects are triples (c, d, α) with c an object of $\mathbf{C}$, d an object of $\mathbf{D}$, and $\alpha : u(c) \xrightarrow{\sim} d$ an isomorphism in $\mathbf{D}$. Morphisms are defined by

$$\mathrm{Hom}_{\mathbf{C}'}((c, d, \alpha), (c', d', \alpha)) = \mathrm{Hom}_{\mathbf{C}}(c, c').$$

Exercise 2.3.3. *Any functor $u : \mathbf{C} \to \mathbf{D}$ factors as a composite*

$$\mathbf{C} \xrightarrow{v} \mathrm{R}(u) \xrightarrow{w} \mathbf{D},$$

where v is a trivial cofibration and w is a fibration.

All this shows:

Theorem 2.3.4. *The category* Cat *admits a model structure, where the weak equivalences, cofibrations, and fibrations are as defined above.*

Remark 2.3.5. This is in fact the *only* model structure on Cat where the weak equivalences are equivalences of categories. The proof is not difficult, see [2].

However there are interesting model structures where the weak equivalences are different, like the Thomason model structure which is equivalent to the Quillen model structure on $\mathrm{Set}_{\boldsymbol{\Delta}}$.

References

[1] Denis-Charles Cisinski, *Higher category theory and homotopical algebra*, Lecture notes, 2016, Available at `http://www.mathematik.uni-regensburg.de/cisinski/CatLR.pdf`.

[2] Chris Schommer-Pries, *The canonical model structure on Cat*, Blog post, 2012, Available at `http://sbseminar.wordpress.com/2012/11/16/the-canonical-model-structure-on-cat/`.