Exercise sheet 4

1. Geometric realization

1.1. Let Top denote the category of compactly generated Hausdorff spaces, and let $r : \mathbf{\Delta} \to \mathrm{Top}$ denote the functor sending $[n]$ to the topological n-simplex Δ^n_{Top}. By the universal property of presheaves, this extends uniquely to a functor

$$|-| := r_! : \mathrm{Set}_{\mathbf{\Delta}} \to \mathrm{Top} \tag{1.1}$$

with the following properties:

(i) It commutes with colimits.

(ii) It sends $\Delta^n \mapsto |\Delta^n| = \Delta^n_{\mathrm{Top}}$.

(iii) It is left adjoint to the restriction functor

$$\mathrm{Sing} := r^* : \mathrm{Top} \to \mathrm{Set}_{\mathbf{\Delta}}.$$

For a simplicial set X, we call the space $|\mathrm{X}|$ the *geometric realization* of X. It is computed by the formula

$$|\mathrm{X}| = \varinjlim_{\Delta^n \to \mathrm{X}} \Delta^n_{\mathrm{Top}}.$$

For a topological space T, we call the simplicial set Sing(T) the *singular complex* of T. By definition, the n-simplices of Sing(T) are topological n-simplices of T:

$$\mathrm{Sing}(\mathrm{T})_n = \mathrm{Hom}_{\mathrm{Top}}(\Delta^n_{\mathrm{Top}}, \mathrm{T}).$$

1.2. Our first goal is to show that the functor $\mathrm{X} \mapsto |\mathrm{X}|$ factors through the category of CW-complexes:

Proposition 1.2.1. *For any simplicial set* X*, the geometric realization* $|\mathrm{X}|$ *is a CW-complex.*

To prove this, recall that X can be written as a filtered colimit of its n-skeleta $\mathrm{Sk}^n(\mathrm{X})$, where each n-skeleton can be built inductively as the push-out:

$$\begin{array}{ccc} \coprod \partial\Delta^n & \hookrightarrow & \coprod \Delta^n \\ \downarrow & & \downarrow{\scriptstyle x} \\ \mathrm{Sk}^{n-1}(\mathrm{X}) & \hookrightarrow & \mathrm{Sk}^n(\mathrm{X}), \end{array}$$

where the coproducts are indexed over the set of non-degenerate n-simplices x.

We would like to define a CW-structure on $|\mathrm{X}|$ with n-skeleta $|\mathrm{X}|^n := |\mathrm{Sk}^n(\mathrm{X})|$. Since $\mathrm{X} \mapsto |\mathrm{X}|$ commutes with colimits, we only need to check:

Exercise 1.2.2. *For each* $n \geqslant 0$ *there are canonical homeomorphisms*

$$|\partial\Delta^n| \approx \partial\Delta^n_{\mathrm{Top}},$$

where $\partial\Delta^n_{\mathrm{Top}}$ *denotes the boundary of the topological* n*-simplex.*

Hint: express $\partial\Delta^n$ as a coequalizer of a diagram involving only terms of the form Δ^k.

Remark 1.2.3. Another proof uses the fact that $\mathrm{X} \mapsto |\mathrm{X}|$ is *left-exact*, i.e. commutes with finite limits. This is rather involved to prove (see Gabriel–Zisman, III.3), but it implies in particular that $\mathrm{X} \mapsto |\mathrm{X}|$ commutes with the operation of taking the image of a morphism.

Similarly:

Exercise 1.2.4. *For each $n \geqslant 0$ and $0 \leqslant k \leqslant n$, there are canonical homeomorphisms*

$$|\Lambda^n_k| \approx (\Lambda^n_k)_{\mathrm{Top}},$$

where $(\Lambda^n_k)_{\mathrm{Top}}$ denotes the topological (n,k)-horn. Further, these are compatible with the inclusions $\Lambda^n_k \hookrightarrow \Delta^n$ in the sense that there are commutative squares

$$\begin{array}{ccc} |\Lambda^n_k| & \xrightarrow{\approx} & (\Lambda^n_k)_{\mathrm{Top}} \\ \big\downarrow & & \big\downarrow \\ |\Delta^n| & \xrightarrow{\approx} & \Delta^n_{\mathrm{Top}} \end{array}$$

2. Homotopy theory of topological spaces (I)

2.1. Given topological spaces $\mathrm{X}, \mathrm{Y} \in \mathrm{Top}$, we say that a continuous map $f : \mathrm{X} \to \mathrm{Y}$ is a *Serre fibration* if it has the right lifting property with respect to the set of topological horn inclusions

$$(\Lambda^n_k)_{\mathrm{Top}} \hookrightarrow \Delta^n_{\mathrm{Top}} \qquad (n \geqslant 0, 0 \leqslant k \leqslant n).$$

Remark 2.1.1. Note that Δ^n_{Top} is homeomorphic to the n-cube, and under this homeomorphism, the (n,k)-horn is identified with the inclusion of the union of the ith faces $(i \neq k)$. Thus f is a Serre fibration iff it has the right lifting property with respect to the inclusions $\mathrm{I}^{n-1} \hookrightarrow \mathrm{I}^n$ (where I denotes the interval $[0,1]$).

Exercise 2.1.2. *A map $f : \mathrm{X} \to \mathrm{Y}$ is a Serre fibration iff the induced morphism* $\mathrm{Sing}(\mathrm{X}) \to \mathrm{Sing}(\mathrm{Y})$ *is a Kan fibration of simplicial sets.*

2.2. We say that a continuous map $f : \mathrm{X} \to \mathrm{Y}$ is a *weak homotopy equivalence* if the the induced maps

$$f_* : \pi_0(\mathrm{X}) \to \pi_0(\mathrm{Y})$$

and

$$f_* : \pi_i(\mathrm{X}, x) \to \pi_i(\mathrm{Y}, f(x))$$

are bijective for all $i > 0$ and all points $x \in \mathrm{X}$.

Exercise 2.2.1. *A map $f : \mathrm{X} \to \mathrm{Y}$ is a weak homotopy equivalence iff the induced morphism* $\mathrm{Sing}(\mathrm{X}) \to \mathrm{Sing}(\mathrm{Y})$ *is a weak homotopy equivalence of simplicial sets.*

Hint: recall that a weak homotopy equivalence of simplicial sets is defined analogously to the above, so it suffices to show that there are functorial bijections $\pi_i(\mathrm{X}, x) \approx \pi_i(\mathrm{Sing}(\mathrm{X}), x)$ for all $x \in \mathrm{X}$.

Exercise 2.2.2. *Let $f : \mathrm{X} \to \mathrm{Y}$ be a Serre fibration. Then f is a weak homotopy equivalence if and only if it has the right lifting property with respect to the inclusions $\partial\Delta^n_{\mathrm{Top}} \to \Delta^n_{\mathrm{Top}}$ for all $n \geqslant 0$.*

References

[1] Denis-Charles Cisinski, *Higher category theory and homotopical algebra*, Lecture notes, 2016, Available at `http://www.mathematik.uni-regensburg.de/cisinski/CatLR.pdf`.

[2] Peter Gabriel and Michel Zisman, *Calculus of fractions and homotopy theory.*