Exercise sheet 5

1. Right, left, and inner fibrations

1.1. Let $p : \mathbf{X} \to \mathbf{C}$ be a functor between two ordinary categories.

Exercise 1.1.1. *The following conditions are equivalent:*

(i) *For any morphism $f : c_0 \to c_1$ in $\mathbf{C}$, and any object x_1 in $\mathbf{X}$ with $p(x_1) = c_1$, there exists a* unique *morphism $\tilde{f} : x_0 \to x_1$ in $\mathbf{X}$ with $p(\tilde{f}) = f$.*

(ii) *For any commutative square*

$$\begin{array}{ccc} \Lambda^1_1 & \longrightarrow & \mathrm{N}(\mathbf{X}) \\ \big\downarrow & & \big\downarrow \\ \Delta^1 & \longrightarrow & \mathrm{N}(\mathbf{C}) \end{array}$$

there exists a unique *lift.*

(iii) *For every integer $n \geqslant 1$ and $0 < k \leqslant n$, and any commutative square*

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & \mathrm{N}(\mathbf{X}) \\ \big\downarrow & & \big\downarrow \\ \Delta^n & \longrightarrow & \mathrm{N}(\mathbf{C}) \end{array}$$

there exists a unique *lift.*

If p satisfies the equivalent conditions above, it is called a *discrete cartesian fibration.*

For a discrete cartesian fibration $p : \mathbf{X} \to \mathbf{C}$, the *fibre* of p over an object $c \in \mathbf{C}$ is the category $\mathbf{X}_c := \mathbf{X} \times_{\mathbf{C}} \{c\}$. The adjective "discrete" is explained by the following:

Exercise 1.1.2. *Every fibre $\mathbf{X}_c$ is discrete (i.e., has no non-identity morphisms).*

Exercise 1.1.3. *Let* $\mathrm{F} : \mathbf{C}^{\mathrm{op}} \to \mathrm{Set}$ *be a presheaf of sets. The category of elements of* F [1, Def. 1.7], *denoted* $\mathbf{C}/\mathrm{F}$, *defines a discrete cartesian fibration $p : \mathbf{C}/\mathrm{F} \to \mathbf{C}$. Further, the assignment* $\mathrm{F} \mapsto \mathbf{C}/\mathrm{F}$ *defines an equivalence of categories between presheaves on $\mathbf{C}$ and discrete cartesian fibrations over $\mathbf{C}$.*

There are generalizations of the above construction to presheaves valued in groupoids (which correspond to "cartesian fibrations in groupoids") or categories (which correspond to "cartesian fibrations").

Dualizing the above discussion, there is a notion of (discrete) cocartesian fibrations, corresponding to covariant functors on $\mathbf{C}$.

1.2. Let $p : \mathbf{X} \to \mathbf{C}$ be a morphism of simplicial sets. Recall [1, Def. 14.1] that p is a *right fibration* if it satisfies the weakening of condition (iii) in Exercise 1.1.1 where the lift is not required to be unique. Dually it is a *left fibration* if it satisfies the right lifting property with respect to the horn inclusions $\Lambda^n_k \hookrightarrow \Delta^n$ for $n \geqslant 1$ and $0 \leqslant k < n$.

Exercise 1.2.1. *For any right (resp. left) fibration $p : \mathbf{X} \to \mathbf{C}$, the fibres are Kan complexes.*

As above, the fibre of p over any 0-simplex $c : \Delta^0 \to \mathbf{C}$ is defined as $\mathbf{X}_c := \mathbf{X} \times_{\mathbf{C},c} \Delta^0$.

1.3. Let X be a simplicial set. Define a bisimplicial set $\mathbf{S}(\mathrm{X})$ by the formula

$$\mathbf{S}(\mathrm{X})_{m,n} := \mathrm{Hom}_{\mathrm{Set}_{\boldsymbol{\Delta}}}((\Delta^m)^{\mathrm{op}} * \Delta^n, \mathrm{X}) \qquad (m, n \geqslant 0).$$

Exercise 1.3.1. *There is a canonical map of bisimplicial sets*

$$\mathbf{S}(\mathrm{X}) \to \mathrm{X}^{\mathrm{op}} \boxtimes \mathrm{X}. \tag{1.1}$$

Show that, if X *is an* ∞*-category, then this map has the right lifting property with respect to the morphisms*

$$\Delta^m \boxtimes \partial\Delta^n \cup \Lambda^m_k \boxtimes \Delta^n \to \Delta^m \boxtimes \Delta^n$$

for $m \geqslant 1, n \geqslant 0$ *and* $0 \leqslant k < m$*, and for* $m \geqslant 0, n \geqslant 1$ *and* $0 \leqslant k < n$*.*

Here $\boxtimes$ denotes the exterior product of bisimplicial sets, see [1, 28.1].

Remark 1.3.2. This exercise says that (1.1) is a *left bifibration*, a notion that will be introduced in the lectures soon.

The interest in this construction is as follows. We can form the diagonal of the bisimplicial set $\mathbf{S}(\mathrm{X})$, denoted $\mathrm{S}(\mathrm{X}) := \delta^*(\mathbf{S}(\mathrm{X}))$. As we will see in the lectures, a corollary of the above is that there is a canonical left fibration $\mathrm{S}(\mathrm{X}) \to \mathrm{X}^{\mathrm{op}} \times \mathrm{X}$. In particular, S(X) is an ∞-category, called the *twisted diagonal* of X, which is very close to the *twisted arrow category* (one has $\mathrm{S}(\mathrm{X}^{\mathrm{op}})^{\mathrm{op}} \approx \mathrm{TwArr}(\mathrm{X})$ in the notation of [2]). This ∞-category will play an important role in the lectures.

1.4. Recall that a morphism of simplicial sets is an *inner fibration* if it has the right lifting property with respect to inner horn inclusions $\Lambda^n_k \hookrightarrow \Delta^n$ ($n \geqslant 2$, $0 < k < n$).

Exercise 1.4.1. *Let* $\mathbf{C}$ *be an ordinary category. For any* ∞*-category* $\mathbf{X}$ *and any morphism* $p : \mathbf{X} \to \mathbf{C}$*,* p *is an inner fibration.*

In particular, any functor of ordinary categories induces an inner fibration on nerves, and so the notion of inner fibration has no 1-categorical counterpart.

Exercise 1.4.2. *A morphism* $p : \mathbf{X} \to \mathbf{C}$ *of simplicial sets is an inner fibration if and only if, for every* n*-simplex* $c : \Delta^n \to \mathbf{C}$*, the fibre* $\mathbf{X}_c := \mathbf{X} \times_{\mathbf{C},c} \Delta^n$ *is an* ∞*-category.*

References

[1] Denis-Charles Cisinski, *Higher category theory and homotopical algebra*, Lecture notes, 2016, Available at `http://www.mathematik.uni-regensburg.de/cisinski/CatLR.pdf`.

[2] Jacob Lurie, *Higher topos theory.*