Exercise sheet 6

1. Categories of correspondences

1.1. Recall that $\mathrm{S}(\mathrm{X})$ denotes the *twisted diagonal* of a simplicial set X, as in Sheet 5.

Exercise 1.1.1. *Describe the ∞-category $\mathrm{S}(\Delta^n)$ for $n \in \mathbf{\Delta}$ (hint: it is the nerve of an ordinary category).*

Below, it will be important to understand the shape of diagrams $\mathrm{S}(\Delta^n) \to \mathbf{C}$.

1.2. Let $\mathbf{C}$ be an ∞-category with two subcategories $\mathbf{C}^{\mathrm{coadm}}$ and $\mathbf{C}_{\mathrm{adm}}$, with the following properties:

(i) Any isomorphism of $\mathbf{C}$ is contained in $\mathbf{C}^{\mathrm{coadm}}$ and $\mathbf{C}_{\mathrm{adm}}$. (Thus in particular each subcategory contains all objects of $\mathbf{C}$.)

(ii) For any morphism in $\mathbf{C}^{\mathrm{coadm}}$ (resp. in $\mathbf{C}_{\mathrm{adm}}$), the base change along a morphism in $\mathbf{C}_{\mathrm{adm}}$ (resp. in $\mathbf{C}^{\mathrm{coadm}}$) exists and belongs to $\mathbf{C}^{\mathrm{coadm}}$ (resp. to $\mathbf{C}_{\mathrm{adm}}$).

We will refer to a morphism of $\mathbf{C}$ as *admissible* (resp. *co-admissible*) if it belongs to the subcategory $\mathbf{C}^{\mathrm{coadm}}$ (resp. $\mathbf{C}_{\mathrm{adm}}$). We will refer to a cartesian square in $\mathbf{C}$ as *bi-admissible* if it is of the form

$$\begin{tikzcd} c_{1,1} \arrow[r, "f'"] \arrow[d, "g'"] & c_{1,0} \arrow[d, "g"] \\ c_{0,1} \arrow[r, "f"] & c_{0,0}, \end{tikzcd}$$

where f and f' are admissible, and g and g' are co-admissible.

1.3. Define a simplicial set

$$\mathrm{Corr}(\mathbf{C}) \in \mathrm{Set}_{\mathbf{\Delta}}$$

whose n-simplices are functors $\delta : \mathrm{S}(\Delta^n)^{\mathrm{op}} \to \mathbf{C}$, satisfying the condition that each commutative square $(0 \leqslant i \leqslant k \leqslant \ell \leqslant j \leqslant n)$

$$\begin{tikzcd} \mathbf{C}_{i,j} \arrow[r] \arrow[d] & \mathbf{C}_{k,j} \arrow[d] \\ \mathbf{C}_{i,\ell} \arrow[r] & \mathbf{C}_{k,\ell}, \end{tikzcd}$$

appearing in the diagram δ, is cartesian and bi-admissible.

Exercise 1.3.1. *Describe the simplicial set $\mathrm{Corr}(\mathbf{C})$ in the case where $\mathbf{C}$ is the nerve of an ordinary category.*

Exercise 1.3.2. *Describe the category $\tau(\mathrm{Corr}(\mathbf{C}))$, where $\mathbf{C}$ is an ∞-category.*

Exercise 1.3.3. *Identify the opposite simplicial set $\mathrm{Corr}(\mathbf{C})^{\mathrm{op}}$.*

1.4. One would like to prove, among other things:

Proposition 1.4.1. *The simplicial set $\mathrm{Corr}(\mathbf{C})$ is an ∞-category.*

This property, as well as most other properties of interest, follow from the following more general statement.

Let $(\mathbf{C}, \mathbf{C}_{\mathrm{adm}}, \mathbf{C}^{\mathrm{coadm}})$ and $(\mathbf{D}, \mathbf{D}_{\mathrm{adm}}, \mathbf{D}^{\mathrm{coadm}})$ be two triples as above. Let $p : \mathbf{C} \to \mathbf{D}$ be a functor which preserves admissible morphisms (resp. co-admissible morphisms) and bi-admissible cartesian squares. First of all, note:

Exercise 1.4.2. *The functor p induces a canonical functor*

$$p : \mathrm{Corr}(\mathbf{C}) \to \mathrm{Corr}(\mathbf{D}). \tag{1.1}$$

The fundamental fact about the construction $\mathbf{C} \mapsto \mathrm{Corr}(\mathbf{C})$ is:

Proposition 1.4.3 (Barwick)**.** *If p is an inner fibration, then (1.1) is an inner fibration.*

This fact is out of our reach, but we have enough technology to check:

Exercise 1.4.4. *If p is a Kan fibration, then (1.1) is a Kan fibration.*

References

[1] Denis-Charles Cisinski, *Higher category theory and homotopical algebra*, Lecture notes, 2016–7, Available at `http://www.mathematik.uni-regensburg.de/cisinski/CatLR.pdf`.

[2] Clark Barwick, *Spectral Mackey functors and equivariant algebraic K-theory (I)*.