Exercise sheet 7

1. Fully faithful functors

1.1. Recall that a functor of ∞-categories $u : \mathrm{X} \to \mathrm{Y}$ is called *fully faithful* if, for each pair of objects x, x' of X, the canonical map

$$\mathrm{X}(x, x') \to \mathrm{Y}(u(x), u(x')) \tag{1.1}$$

is an equivalence of ∞-groupoids.

Exercise 1.1.1. *Show that $u : \mathrm{X} \to \mathrm{Y}$ is fully faithful if and only if the canonical map*

$$\mathrm{S}(\mathrm{X}) \to (\mathrm{X}^{\mathrm{op}} \times \mathrm{X}) \underset{\mathrm{Y}^{\mathrm{op}} \times \mathrm{Y}}{\times} \mathrm{S}(\mathrm{Y}) \tag{1.2}$$

is a fibre-wise equivalence over $\mathrm{X}^{\mathrm{op}} \times \mathrm{X}$, where $\mathrm{S}(-)$ denotes the twisted diagonal.

1.2. Given a simplicial set K, let $\mathrm{S} \subset \mathrm{K}_0$ be a subset of 0-simplices. Define a simplicial subset $\mathrm{K}_\mathrm{S} \subset \mathrm{K}$ whose n-simplices are maps $\Delta^n \to \mathrm{K}$ which send each vertex of Δ^n to a 0-simplex in $\mathrm{S} \subset \mathrm{K}_0$.

Exercise 1.2.1.

(i) *The inclusion $\mathrm{K}_\mathrm{S} \hookrightarrow \mathrm{K}$ has the right lifting property with respect to any map of simplicial sets $\mathrm{A} \to \mathrm{B}$ which induces a bijection $\mathrm{A}_0 \to \mathrm{B}_0$ on 0-simplices.*

(ii) *Suppose that K is an ∞-category. Then K_S is an ∞-category, and the map $\mathrm{K}_\mathrm{S} \hookrightarrow \mathrm{K}$ is a fully faithful functor.*

(iii) *Suppose that for any isomorphism $x \to y$ in K, the 0-simplex x belongs to S iff y does. Then the inclusion $\mathrm{K}_\mathrm{S} \hookrightarrow \mathrm{K}$ is further an isofibration.*

1.3. Let $u : \mathrm{X} \to \mathrm{Y}$ be a functor. Let $\mathrm{S} \subset \mathrm{Y}_0$ be the subset of 0-simplices which are in the essential image of u, and consider the simplicial subset $\mathrm{Y}_\mathrm{S} \subset \mathrm{Y}$ as above.

Exercise 1.3.1. *The functor u is fully faithful if and only if the induced functor*

$$u_\mathrm{S} : \mathrm{X} \to \mathrm{Y}_\mathrm{S}$$

is an equivalence of ∞-categories.

Exercise 1.3.2. *The functor $u : \mathrm{X} \to \mathrm{Y}$ is fully faithful if and only if there exists a factorization*

$$\mathrm{X} \xrightarrow{v} \mathrm{X}' \xrightarrow{w} \mathrm{Y},$$

where v is an equivalence of ∞-categories and w has the right lifting property with respect to the boundary inclusions $\partial\Delta^n \hookrightarrow \Delta^n$ for $n \geqslant 1$.

Hint: factor u into a trivial cofibration, followed by a fibration in the Joyal model structure.

2. Final objects

2.1. Recall that an object x of an ∞-category X is *final* if the map $x : \Delta^0 \to \mathrm{X}$ is final. We have seen that this is equivalent to the condition that the ∞-groupoid $\mathrm{X}(a, x)$ is contractible for all objects a in X.

Exercise 2.1.1. *Let X be an ∞-category and consider the full subcategory $\mathrm{X}_{\mathrm{final}}$ of final objects of X. Then $\mathrm{X}_{\mathrm{final}}$ is an ∞-groupoid which is empty or contractible.*

2.2. Let **S** denote the ∞-category of left fibrations with **U**-small fibres.

Exercise 2.2.1. *The ∞-category **S** admits a final object.*