Lecture 4
Zariski descent for algebraic K-theory

Our goal for this lecture is to demonstrate Zariski descent for algebraic K-theory.

1. Algebraic K-theory.

1.1. Let $\mathcal{C}$ be a small stable $\infty$-category. We say that $\mathcal{C}$ is idempotent-complete if the canonical functor

$$\mathcal{C} \to (\mathcal{C})^{\text{idem}}$$

is an equivalence. Recall that $(\mathcal{C})^{\text{idem}}$, the idempotent-completion, can be defined as the full subcategory of presheaves on $\mathcal{C}$ generated by the representables under direct summands; equivalently it is the full subcategory of compact objects in the ind-completion: $(\mathcal{C})^{\text{idem}} = \text{Ind}(\mathcal{C})^{\text{comp}}$.

In particular, for any stable presentable $\infty$-category $\mathcal{C}$, the full subcategory $(\mathcal{C})^{\text{comp}}$ is idempotent-complete.

1.2. Nonconnective algebraic K-theory defines a functor

$$(1.1) \quad K : \text{Stab}^{\text{idem}} \to \text{Spt}$$

from the $\infty$-category of small stable idempotent-complete $\infty$-categories (and exact functors) to the $\infty$-category of spectra.

This functor has two especially important properties:

(i) $K$ preserves filtered colimits.

(ii) For any exact sequence of small stable idempotent-complete $\infty$-categories $\mathcal{C}' \to \mathcal{C} \to \mathcal{C}''$, the induced sequence of spectra

$$K(\mathcal{C}') \to K(\mathcal{C}) \to K(\mathcal{C}'')$$

is an exact triangle.

Instead of recalling the construction of the functor $K$, we will instead take an axiomatic approach. The idea is that the arguments we make will actually apply to a large class of interesting functors.

Definition 1.3. Let $E : \text{Stab} \to \text{Spt}$ be a functor. We say that it is continuous if it satisfies (i) and localizing if it satisfies (iii).

Example 1.4. Nonconnective K-theory is continuous and localizing; see [2]. The reader can also find a construction of the functor $K$ in loc. cit.

Remark 1.5. By applying the “connective cover” functor to $K : \text{Stab}^{\text{idem}} \to \text{Spt}$, we obtain the connective K-theory functor $K^{\text{cn}} : \text{Stab}^{\text{idem}} \to \text{Spt}^{\text{cn}}$. Connective K-theory fails to be localizing; instead it satisfies a weaker property called additivity: it sends any split exact sequence in $\text{Stab}^{\text{idem}}$ to a split exact sequence of spectra.

On the other hand, $\text{Spt} \to \text{Spt}^{\text{cn}}$ commutes with limits, so any descent property we show for nonconnective K-theory will give us a descent result for connective K-theory (as a presheaf of connective spectra).

1.6.
2. The localization sequence in K-theory. The discussion in this section will apply to any localizing functor $E$ instead of $K$.

2.1. Given a quasi-compact open immersion $j : U \hookrightarrow X$, we write

$$K(X)_U = K(\text{Perf}(X)_U).$$

Recall from Lecture 3 that $\text{Perf}(X)_U$ denotes the full subcategory of perfect complexes on $X$ which vanish on $U$.

**Theorem 2.2** (Thomason). Let $j : U \hookrightarrow X$ be a quasi-compact open immersion of qcqs derived schemes. Then there is a canonical exact triangle

$$K(X)_U \to K(X) \to K(U)$$

of spectra.

**Proof.** This follows from the exactness of the sequence

$$\text{Perf}(X)_U \hookrightarrow \text{Perf}(X) \to \text{Perf}(U),$$

which we saw in Lecture 3, in view of the localizing property of the functor $K : \text{Stab}^{\text{idem}} \to \text{Spt}$. \qed

**Remark 2.3.** When the schemes are classical and regular (nonsingular), we can identify the fibre term $K(X)_U$ more explicitly; we will come back to this later this lecture.

3. Zariski descent in K-theory. As in the previous section, we can replace $K$ by any localizing functor $E$ in this section as well.

3.1. We have:

**Theorem 3.2** (Thomason). Let $X$ be a qcqs derived scheme and let $X = U \cup V$ be a Zariski open cover. Then the induced square of spectra

$$
\begin{array}{ccc}
K(X) & \longrightarrow & K(U) \\
\downarrow & & \downarrow \\
K(V) & \longrightarrow & K(U \cap V)
\end{array}
$$

is cartesian.

**Remark 3.3.** We will not make this explicit here, but it follows from a theorem of Voevodsky [9] that on the full subcategory $\text{DSch}_{\text{qcqs}}$ of quasi-compact quasi-separated derived schemes, the above condition is equivalent to Čech descent with respect to the Zariski topology for the presheaf of spectra $K : (\text{DSch}_{\text{qcqs}})^{\text{op}} \to \text{Spt}$.

**Proof.** The claim is that the canonical map

$$\delta : K(X) \to K(U) \times_{K(U \cap V)} K(V)$$

is invertible. It suffices to show that the map induced on the fibres,

$$\varepsilon : \text{Fib}(K(X) \to K(U)) \to \text{Fib}(K(V) \to K(U \cap V)),$$
is invertible. Indeed, write \( F := K(U) \times_{K(U \cap V)} K(V) \) and consider the diagram of cartesian squares

\[
\begin{array}{ccc}
\text{Fib}(K(X) \to K(U)) & \to & K(X) \\
\downarrow^\varepsilon & & \downarrow^\delta \\
\text{Fib}(K(V) \to K(U \cap V)) & \to & F \\
\downarrow & & \downarrow \\
0 & \to & K(U) & \to & K(U \cap V).
\end{array}
\]

By stability of the \( \infty \)-category of spectra, each of these squares, in particular the upper one, is also cocartesian.

By the localization sequence the map \( \varepsilon \) is identified with the canonical map

\[ K(X)_U \to K(V)_{U \cap V} \]

which is induced by the canonical functor

\[ \text{Perf}(X)_U \to \text{Perf}(V)_{U \cap V}, \]

which is an equivalence by Zariski excision for the presheaf of \( \infty \)-categories \( X \mapsto \text{Perf}(X) \) (Lecture 3).

\( \square \)

4. Coherent sheaves and \( G \)-theory. When the schemes are classical and regular (nonsingular), one can identify the fibre term in the localization sequence much more explicitly.

4.1. First we define coherent sheaves in the derived setting. The definition is simpler when we impose a finiteness condition on the schemes.

**Definition 4.2.** Let \( R \) be a simplicial commutative ring. We say that \( R \) is coherent if the following hold:

(i) \( \pi_0(R) \) is coherent in the ordinary sense, i.e. every finitely generated ideal is finitely presented.

(ii) For each \( i \), the \( \pi_0(R) \)-module \( \pi_i(R) \) is of finite presentation.

We say that \( R \) is noetherian if it is coherent and \( \pi_0(R) \) is noetherian in the ordinary sense (i.e. every ideal is finitely generated).

The following is a generalization of the notion of “pseudocoherence” from SGA 6.

**Definition 4.3.** Let \( R \) be a coherent simplicial commutative ring. An \( R \)-module \( M \) is almost perfect if the following hold:

(i) \( M \) is eventually connective, i.e. there exists some integer \( i \) such that \( \pi_n(M) = 0 \) for all \( n < i \).

(ii) For each \( i \), the \( \pi_0(R) \)-module \( \pi_i(M) \) is of finite presentation.

**Exercise 4.4.** The property of almost perfectness is stable under finite (co)limits and direct summands.

**Corollary 4.5.** Let \( R \in \text{SCRing} \) be coherent. Then any perfect \( R \)-module is almost perfect.

**Proof.** Since \( R \) itself is almost perfect as an \( R \)-module, this follows from Exercise 4.4. \( \square \)

**Remark 4.6.** One can show that an \( R \)-module \( M \) is perfect iff it is almost perfect and of finite tor-amplitude.

**Remark 4.7.** One can define almost perfectness without the coherence assumption on \( R \); see [4, § 7.2.4].
Definition 4.8. Let $R$ be a coherent simplicial commutative ring. An $R$-module $M$ is coherent if it is almost perfect and eventually coconnective, i.e. there exists some integer $i$ such that $\pi_n(M) = 0$ for all $n > 0$.

We let $\text{Mod}^{\text{coh}}_R$ denote the full subcategory of $\text{Mod}_R$ spanned by coherent $R$-modules. This is a stable idempotent-complete subcategory.

Remark 4.9. Let $R$ be an ordinary commutative ring. Then we can think of $M$ as a cochain complex of (ordinary) $R$-modules, and coherence amounts to the condition that it is bounded (above and below), and its cohomologies $H^n(M)$ are finitely presented $H^0(R)$-modules. Thus $\text{Mod}^{\text{coh}}_R$ is equivalent to the bounded derived category of coherent sheaves on $\text{Spec}(R)$ in the usual sense.

Remark 4.10. Unlike in the classical setting, there is no inclusion $\text{Mod}^{\text{perf}}_R \subset \text{Mod}^{\text{coh}}_R$ in general. Indeed, $R$ itself may not be eventually coconnective.

If we suppose that $R$ is eventually coconnective, then any perfect $R$-module $M$ is eventually coconnective, since the latter property is stable under finite (co)limits and direct summands. In this case we do have an inclusion $\text{Mod}^{\text{perf}}_R \subset \text{Mod}^{\text{coh}}_R$.

4.11. We now globalize the above definitions.

Definition 4.12. Let $X$ be a derived scheme. We say that $X$ is locally coherent if for any affine derived scheme $S = \text{Spec}(R)$ and any open immersion $j : S \hookrightarrow X$, the simplicial commutative ring $R$ is coherent. We say that $X$ is coherent if it is locally coherent and quasi-compact.

Given a locally coherent derived scheme $X$, we say that a quasi-coherent sheaf $\mathcal{F} \in \text{Qcoh}(X)$ is coherent if for any affine derived scheme $S = \text{Spec}(R)$ and any open immersion $j : S \hookrightarrow X$, the inverse image $j^* \mathcal{F}$ is coherent. We let $\text{Coh}(X) \subset \text{Qcoh}(X)$ denote the full subcategory spanned by coherent sheaves. By the discussion above, this is an idempotent-complete stable small $\infty$-category.

4.13. Let $X$ be a locally coherent derived scheme.

Definition 4.14. The $G$-theory of $X$ is defined as the spectrum

$$G(X) = K(\text{Coh}(X)).$$

For a classical noetherian scheme $X$, a theorem of Schlichting [6] implies:

Theorem 4.15. Let $X$ be a noetherian classical scheme. Then the spectrum $G(X)$ is connective.

If we suppose further that $X$ is regular (nonsingular), then one can show that the inclusion $\text{Perf}(X) \subset \text{Coh}(X)$ is an equivalence [3, Exp. I]. Therefore, we have:

Proposition 4.16. Let $X$ be a regular noetherian classical scheme. Then the canonical map of spectra

$$K(X) \to G(X)$$

is an equivalence.

Corollary 4.17. Let $X$ be a regular noetherian classical scheme. Then $K(X)$ is connective, i.e. the canonical map of spectra

$$K^{\text{cn}}(X) \to K(X)$$

is an equivalence.
4.18. Quillen’s dévissage shows:

**Theorem 4.19** (Quillen). Let \( X \) be a noetherian classical scheme, \( j : U \hookrightarrow X \) an open immersion, and \( i : Z \hookrightarrow X \) a complementary closed immersion. Then we have an exact triangle of spectra
\[
G(Z) \to G(X) \to G(U).
\]

In particular, if \( X \) is regular, then we get \( K(X)_U = G(Z) \) in this situation.

4.20. For derived schemes, the relation between regularity and an isomorphism \( K(X) \simeq G(X) \) is more subtle. As pointed out above, there is not even an inclusion \( \text{Perf}(X) \subset \text{Coh}(X) \) in general, and hence no canonical map \( K(X) \to G(X) \).

Let \( R \in \text{SCRing} \) be coherent. Say that \( R \) is **almost regular** if any coherent \( R \)-module \( M \) is of finite tor-amplitude, hence perfect. Then by definition, if \( R \) is eventually coconnective and almost regular, we have an equivalence \( \text{Mod}^{\text{perf}}_R = \text{Mod}^{\text{coh}}_R \), and in particular an isomorphism of spectra
\[
K(R) \simeq G(R).
\]

If \( \pi_0(R) \) is regular, it is easy to see that \( R \) is almost regular iff \( \pi_0(R) \) is of finite tor-amplitude as an \( R \)-module. See [1] for further discussion.

**References.**


