Exercise sheet 10

1. Show that for any noetherian ring A and every $n \geqslant 0$, inverse image along $\phi : \mathrm{A} \to \mathrm{A}[\mathrm{T}_1, \ldots, \mathrm{T}_n]$ induces an isomorphism

$$\phi^* : \mathrm{G}_0(\mathrm{A}) \to \mathrm{G}_0(\mathrm{A}[\mathrm{T}_1, \ldots, \mathrm{T}_n]).$$

(We showed this when A is a field and $n = 1$. For the case of an integral domain, use a noetherian induction argument and the localization sequence to reduce to the fraction field.)

2. Let k be an algebraically closed field and $\mathrm{A} = k[\mathrm{X}, \mathrm{Y}, \mathrm{Z}]/\langle \mathrm{XZ}, \mathrm{Z}(\mathrm{Z}^2 - \mathrm{Y}^3)\rangle$. Show that $|\mathrm{Spec}(\mathrm{A})|$ has two irreducible components, $\mathrm{Y}_1 = \mathrm{V}(\langle \mathrm{Z}\rangle)$ and $\mathrm{Y}_2 = \mathrm{V}(\langle \mathrm{X}, \mathrm{Z}^2 - \mathrm{Y}^3\rangle)$, and is not of pure dimension.

3. Let k be an algebraically closed field and $\mathrm{A} = k[\mathrm{X}, \mathrm{Y}]$. Let $f, g \in \mathrm{A}$ be polynomials. In each of the following examples, determine whether $\mathrm{V}(f)$ and $\mathrm{V}(g)$ intersect properly, and compute the cycle $[\mathrm{A}/\langle f, g\rangle]_d \in \mathrm{Z}_d(\mathrm{A})$, where $d = \dim(\mathrm{V}(f)) + \dim(\mathrm{V}(g)) - \dim(\mathrm{A})$.

(a) $f = \mathrm{X}$, $g = \mathrm{Y}$

(b) $f = \mathrm{X}$, $g = \mathrm{X}$

(c) $f = \mathrm{Y} - \mathrm{X}^2$, $g = \mathrm{Y}$

(d) $f = \mathrm{XY}$, $g = \mathrm{Y}^2$

(Note: we only defined properness of intersection between *irreducible* subsets. However the same definition makes sense as long as both subsets are of pure dimension.)

4. Let A be a noetherian ring and $f \in \mathrm{A}$ an element. Let $\phi : \mathrm{A} \to \mathrm{A}[f^{-1}]$ and $\psi : \mathrm{A} \to \mathrm{A}/\langle f\rangle$. Show that there is an exact sequence

$$\mathrm{CH}_n(\mathrm{A}/\langle f\rangle) \xrightarrow{\psi_*} \mathrm{CH}_n(\mathrm{A}) \xrightarrow{\phi^*} \mathrm{CH}_n(\mathrm{A}[f^{-1}]) \to 0$$

for every n.