Exercise sheet 11

1. Let A be a ring.

(a) Let M and N be invertible A-modules. Show that $[\mathrm{M}] = [\mathrm{N}]$ in $\mathrm{K}_0(\mathrm{A})$ iff $\mathrm{M} \simeq \mathrm{N}$ as A-modules.

(b) Let M be a f.g. projective A-module such that $[\mathrm{M}] \in \mathrm{K}_0(\mathrm{A})$ is a unit. Show that M is invertible as an A-module.

(c) Show that the group homomorphism $\mathrm{Pic}(\mathrm{A}) \to \mathrm{K}_0(\mathrm{A})^\times$ is injective but not always bijective.

2. Let A be a regular ring. Show that the homomorphism $\mathrm{Pic}(\mathrm{A}) \to \mathrm{Pic}(\mathrm{A}[\mathrm{T}_1, \ldots, \mathrm{T}_n])$, induced by extension of scalars, is bijective for all $n > 0$.

3. Let A be an integral domain. Recall the rank homomorphism $\mathrm{rk} : \mathrm{G}_0(\mathrm{A}) \to \mathbf{Z}$ (Sheet 3, Exercise 4), which we regard as a homomorphism $\mathrm{rk} : \mathrm{K}_0(\mathrm{A}) \to \mathbf{Z}$ by restricting along the canonical homomorphism $\mathrm{K}_0(\mathrm{A}) \to \mathrm{G}_0(\mathrm{A})$. Let $x \in \mathrm{K}_0(\mathrm{A})$ be a class of positive rank ($\mathrm{rk}(x) > 0$). Show that

$$n \cdot x = [\mathrm{M}]$$

for some $\mathrm{M} \in \mathrm{Mod}_{\mathrm{A}}^{\mathrm{fgproj}}$ and integer $n \geqslant 0$.

Hint: reduce to the case where A is of finite dimension d, and use a theorem of Serre which states that any projective A-module is the direct sum of a free module and a projective module of rank $\leqslant d$ (see [Serre, *Modules projectifs et espaces fibrés à fibre vectorielle*]).

4. Let k be an algebraically closed field and $\mathrm{A} = k[\mathrm{X}, \mathrm{Y}]/\langle \mathrm{X}^2 - \mathrm{Y}^3 \rangle$ (an integral domain of dimension 1). Let $f \in \mathrm{Frac}(\mathrm{A})^\times$ denote the rational function $(\mathrm{X}-\mathrm{Y})/\mathrm{Y}$. For the closed point $x_0 = \mathrm{V}(\langle \mathrm{X}, \mathrm{Y} \rangle)$ in $|\mathrm{Spec}(\mathrm{A})|$, show that $f_{\mathfrak{p}(x_0)} \in \mathrm{Frac}(\mathrm{A}_{\mathfrak{p}(x_0)})$ is not contained in the subring $\mathrm{A}_{\mathfrak{p}(x_0)}$. For every other closed point $x \neq x_0$, show that $f_{\mathfrak{p}(x)}$ is even contained in the subgroup of units $\mathrm{A}_{\mathfrak{p}(x)}^\times$. Deduce that the principal Cartier divisor $\mathrm{div}_{\mathrm{A}}(f) \in \mathrm{Cart}(\mathrm{A})$ is nonzero.