Exercise sheet 13

1. Let k be a field and $i : \mathrm{Z} \to \mathrm{X}$ a closed immersion of affine k-schemes.

(a) Show that i is flat if and only if the ideal

$$\mathrm{I} = \mathrm{Ker}(\Gamma(\mathrm{X}, \mathcal{O}_{\mathrm{X}}) \to \Gamma(\mathrm{Z}, \mathcal{O}_{\mathrm{Z}}))$$

is idempotent, i.e., $\mathrm{I}^2 = \mathrm{I}$.

(b) Assume that X is *connected* and noetherian, i.e., $\Gamma(\mathrm{X}, \mathcal{O}_{\mathrm{X}})$ is a noetherian ring admitting no nontrivial idempotents. Show that in this case, i is flat iff $\mathrm{Z} = \mathrm{X}$ ($\mathrm{I} = \langle 0 \rangle$) or Z is empty ($\mathrm{I} = \langle 1 \rangle$).

2. Let $i_1 : \mathrm{Z}_1 \to \mathrm{X}$ and $i_2 : \mathrm{Z}_2 \to \mathrm{X}$ be closed immersions of affine k-schemes. We say that Z_1 and Z_2 *intersect transversely* in X if the square

$$\begin{array}{ccc} \mathrm{Z}_1 \times_{\mathrm{X}} \mathrm{Z}_2 & \longrightarrow & \mathrm{Z}_2 \\ \downarrow & & \downarrow{\scriptstyle i_2} \\ \mathrm{Z}_1 & \xrightarrow{i_1} & \mathrm{X} \end{array}$$

is Tor-independent. That is, the induced square of commutative rings

$$\begin{array}{ccc} \Gamma(\mathrm{X}, \mathcal{O}_{\mathrm{X}}) & \longrightarrow & \Gamma(\mathrm{Z}_1, \mathcal{O}_{\mathrm{Z}_1}) \\ \downarrow & & \downarrow \\ \Gamma(\mathrm{Z}_2, \mathcal{O}_{\mathrm{Z}_2}) & \longrightarrow & \Gamma(\mathrm{Z}_1, \mathcal{O}_{\mathrm{Z}_1}) \otimes_{\Gamma(\mathrm{X}, \mathcal{O}_{\mathrm{X}})} \Gamma(\mathrm{Z}_2, \mathcal{O}_{\mathrm{Z}_2}) \end{array}$$

is Tor-independent.

Show that the self-intersection of a closed immersion $i : \mathrm{Z} \to \mathrm{X}$ is transverse if and only if i is flat.

3. Let X be a smooth affine k-scheme. Let Y and Z be integral closed subschemes. Suppose that Y and Z intersect transversally in X. Then

$$\gamma[\mathrm{Y}] \cup \gamma[\mathrm{Z}] = \gamma([\mathrm{Y}] \cup [\mathrm{Z}])$$

in $\mathrm{G}_0(\mathrm{X})$. Here $\gamma : \mathrm{Z}_*(\mathrm{X}) \to \mathrm{G}_0(\mathrm{X})$ is the homomorphism $[\mathrm{Y}] \mapsto [\mathcal{O}_{\mathrm{Y}}]$ (see Sheet 9, Exercise 3). The cup product $[\mathrm{Y}] \cup [\mathrm{Z}] \in \mathrm{Z}_*(\mathrm{X})$ is the intersection product defined using the Tor formula as in §10.6.