Exercise sheet 6

1. Let A be a noetherian ring. Show that $G_0(A)$ is generated by classes $[A/\mathfrak{p}]$ where $\mathfrak{p}$ is a prime ideal.

Recall from Lecture 1 that every f.g. A-module M admits a finite filtration whose quotients are A-modules of the form $A/\mathfrak{p}$, with $\mathfrak{p} \subset A$ prime. Then by the Observation in §6.3,

$$[M] = \sum_i [A/\mathfrak{p}_i]$$

where $\{\mathfrak{p}_i\}_i$ is a finite set of prime ideals.

2. Let A be a commutative ring. A chain complex $M_\bullet$ of A-modules is called *formal* if it is quasi-isomorphic to

$$\bigoplus_{i\in\mathbf{Z}} H_i(M_\bullet)[i],$$

the complex with $H_i(M_\bullet)$ in degree i and all differentials zero.

(i) Let k be a field. Show that every chain complex of k-modules is formal.

(ii) Give an example of a non-formal complex over a commutative ring A.

(i) Let $V_\bullet \in \mathrm{Ch}_k$. Write $H_i := H_i(V_\bullet)$ for all $i \in \mathbf{Z}$ for simplicity, and $H_\bullet := \bigoplus_{i\in\mathbf{Z}} H_i[i]$. It will suffice to construct a morphism of chain complexes

$$\phi : V_\bullet \to H_\bullet$$

inducing the identity maps $H_i \to H_i$ on homologies.

Set $Z_i := \mathrm{Ker}(d_i)$ and $B_i := \mathrm{Im}(d_{i+1})$ so that $H_i = Z_i/B_i$ for all i. Since k is a field, both short exact sequences

$$0 \to Z_i \hookrightarrow V_i \twoheadrightarrow B_{i-1} \to 0,$$
$$0 \to B_i \hookrightarrow Z_i \twoheadrightarrow H_n \to 0$$

split, whence isomorphisms

$$V_i \simeq Z_i \oplus B_{i-1} \simeq B_i \oplus H_i \oplus B_{i-1}.$$

Under these isomorphisms, the differential $d_i : V_i \to V_{i-1}$ is induced by $\mathrm{id} : B_{i-1} \to B_{i-1}$ (and zero on the other components).

We define the morphism $\phi : V_\bullet \to H_\bullet$ by the projections

$$V_i \simeq B_i \oplus H_i \oplus B_{i-1} \xrightarrow{\mathrm{pr}} H_i$$

for each i. This is clearly a morphism of chain complexes that induces identity maps on homologies, hence it is a quasi-isomorphism.

(ii) It is somewhat tricky to show that two complexes are not quasi-isomorphic, when they happen to have the same homology groups. One way to proceed is by noting that some properties are invariant under quasi-isomorphism, such as perfectness or finite Tor-amplitude. Thus it will suffice to construct a perfect complex $M_\bullet$ which has at least two nontrivial homology groups, at least one of which is not perfect. This will only be possible if the ring A is not regular (a perfect complex has f.g. homology groups, so if the ring is regular, the homology groups are perfect). Such an $M_\bullet$ will have $H_\bullet := \bigoplus_{i\in\mathbf{Z}} H_i(M_\bullet)[i]$ not perfect, hence there will be no quasi-isomorphism $M_\bullet \simeq H_\bullet$.

Let k be a field and $A = k[\epsilon]/(\epsilon^2)$. Consider the complex of A-modules

$$M_\bullet = \left(0 \to A \xrightarrow{\epsilon} A \to 0\right).$$

Note that $M_\bullet$ is a finite complex of f.g. free modules, so it is perfect. But we have $H_0(M_\bullet) = k$ and from Sheet 2 we know that k is not perfect as an A-module. Thus $H_\bullet$ can not be perfect either. Indeed it will have infinite Tor-amplitude as

$$H_\bullet \otimes^{\mathbf{L}}_A N = \bigoplus_{i\in\mathbf{Z}} H_i(M_\bullet)[i] \otimes^{\mathbf{L}}_A N = \bigoplus_{i\in\mathbf{Z}} \left(H_i(M_\bullet) \otimes^{\mathbf{L}}_A N\right)[i]$$

for any A-module N, and so $H_n(H_\bullet \otimes^{\mathbf{L}}_A N)$ will contain $H_n(H_0(M_\bullet) \otimes^{\mathbf{L}}_A N) \neq 0$ for all $n \geqslant 0$.

3. (i) Let A be a commutative ring and $I \subset A$ an ideal contained in the Jacobson radical of A. Show that the homomorphism $\mathcal{M}(A) \to \mathcal{M}(A/I)$, given by extension of scalars along the quotient map $\phi : A \to A/I$, is injective. Recall that $\mathcal{M}(A)$ denotes the monoid of isomorphism classes of f.g. projective A-modules. (Hint: Nakayama.)

(ii) Suppose that I is a nil ideal, i.e., that every element $x \in I$ is nilpotent. Let $\phi : A \to A/I$ be the quotient map. Show that the homomorphism $\phi^* : K_0(A) \to K_0(A/I)$ is invertible. (Hint: use the fact that idempotents lift along nil ideals in associative rings, and apply this to the algebra of endomorphisms of $A^{\oplus n}$.)

(i) Let M and N be f.g. projective A-modules and suppose there exists a morphism $\overline{\phi} : M/IM \to N/IM$ of A/I-modules. Since M is projective, there exists a morphism $\phi : M \to N$ such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\phi} & N \\ \downarrow & & \downarrow \\ M/IM & \xrightarrow{\overline{\phi}} & N/IN \end{array}$$

commutes. If $\overline{\phi}$ is surjective, then so is ϕ. Equivalently, $Q = \mathrm{Coker}(\phi)$ is zero iff $Q/IQ = 0$; this follows from Nakayama since I is contained in the Jacobson radical and since Q is finitely generated. Now suppose that $\overline{\phi}$ is also injective. Then since M is projective and ϕ is surjective, it admits a section $\sigma : N \to M$ and M splits as a direct sum $K \oplus N$, where $K = \mathrm{Ker}(\phi)$. Now we have $K/IK = 0$ since $\overline{\phi}$ induces an isomorphism $M/IM \simeq N/IN$ by assumption. Thus by Nakayama again (since K is also f.g.), we conclude that $K = 0$. In particular, ϕ is an isomorphism. This discussion shows that every A/I-module isomorphism $M/IM \simeq N/IM$ lifts to an A-module isomorphism $M \simeq N$. In particular, the map $\mathcal{M}(A) \to \mathcal{M}(A/I)$ is injective.

(ii) First of all, note that every nil ideal is contained in the Jacobson radical.

For injectivity, let $x \in K_0(A)$ such that $\phi^*(x) = 0$ in $K_0(A/I)$. By the Lemma in §3.1, we can write $x = [M] - [A^{\oplus n}]$, where $M \in \mathrm{Mod}_A^{\mathrm{fgproj}}$ and $n \geqslant 0$. Then we have $[M/IM] = [(A/I)^{\oplus n}] \in K_0(A/I)$ so by part (ii) of the same Lemma, we deduce that $M/IM \oplus (A/I)^{\oplus k} \simeq (A/I)^{\oplus n+k}$ for some $k \geqslant 0$. Then by claim (i) above, it follows that $M \oplus A^{\oplus k} \simeq A^{\oplus n+k}$ as A-modules, and in particular $[M] = [A^{\oplus n}]$ in $K_0(A)$, i.e., $x = 0$.

For surjectivity, it will suffice to show that every f.g. projective A/I-module N lifts to a f.g. projective A-module M (for which $M \otimes_A A/I = N$). Since N is a direct summand of a f.g. free module $(A/I)^{\oplus n}$, there is a corresponding projector (= idempotent endomorphism)

$$\overline{e} : (A/I)^{\oplus n} \twoheadrightarrow N \hookrightarrow (A/I)^{\oplus n}$$

whose image is N. Applying the hint to the homomorphism of matrix rings

$$\mathrm{End}_A(A^{\oplus n}) \to \mathrm{End}_{A/I}((A/I)^{\oplus n})$$

(which is the quotient by a nil ideal), we deduce that there exists an idempotent endomorphism $e : A^{\oplus n} \to A^{\oplus n}$ lifting $\overline{e}$. The image of e is a f.g. projective A-module M lifting N.

For a proof of the claim in the hint, see e.g. [Bass, Algebraic K-theory, Chap. III, Prop. 2.10].

4. Let $\phi : A \to B$ be a ring homomorphism which exhibits B as a f.g. free A-module of rank d. Then we have $[B] = d.[A] = d$ in $K_0(A)$. Show that the composites

$$K_0(A) \xrightarrow{\phi^*} K_0(B) \xrightarrow{\phi_*} K_0(A)$$

$$K_0(B) \xrightarrow{\phi_*} K_0(A) \xrightarrow{\phi^*} K_0(B)$$

are both given by multiplication by d.

By assumption, B is flat as an A-module. In particular, the square

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \beta} \\ B & \xrightarrow{\alpha} & B \otimes_A B = B' \end{array}$$

is Tor-independent. Thus by the base change formula (§6.2) we have

$$\phi^*\phi_* = \beta_*\alpha^* : K_0(B) \to K_0(B).$$

Take $x = [\mathrm{N}] - [\mathrm{B}^{\oplus n}] \in \mathrm{K}_0(\mathrm{B})$, $\mathrm{N} \in \mathrm{Mod}_{\mathrm{B}}^{\mathrm{fgproj}}$ and $n \geqslant 0$ (recall every x can be written in this form, by §3.1). We have

$$\begin{aligned} \beta_* \alpha^*(x) &= \beta_*([\mathrm{N} \otimes_{\mathrm{B}} \mathrm{B} \otimes_{\mathrm{A}} \mathrm{B}] - [\mathrm{B}^{\oplus n} \otimes_{\mathrm{B}} \mathrm{B} \otimes_{\mathrm{A}} \mathrm{B}]) \\ &= [\mathrm{N} \otimes_{\mathrm{B}} \mathrm{B}] \cup [\mathrm{B}] - [\mathrm{B}^{\oplus n} \otimes_{\mathrm{B}} \mathrm{B}] \cup [\mathrm{B}] \\ &= x \cup d \end{aligned}$$

as desired.

For the composite $\phi_* \phi^*$ we first note

$$\phi_* \phi^* [\mathrm{A}] = \phi_* [\mathrm{B}] = [\mathrm{B}_{[\mathrm{A}]}] = d.$$

Then by the projection formula,

$$d \cup x = (\phi_* \phi^* [\mathrm{A}]) \cup x = \phi_* (\phi^* [\mathrm{A}] \cup \phi^*(x)) = \phi_* \phi^* ([\mathrm{A}] \cup x) = \phi_* \phi^*(x).$$

(Recall that ϕ^* is a ring homomorphism and that $[\mathrm{A}]$ is the unit of the ring structure on $\mathrm{K}_0(\mathrm{A})$.)