**Exercise sheet 6**

**1.** Let A be a noetherian ring. Show that $G_0(A)$ is generated by classes $[A/\mathfrak{p}]$ where $\mathfrak{p}$ is a prime ideal.

**2.** Let A be a commutative ring. A chain complex $M_\bullet$ of A-modules is called *formal* if it is quasi-isomorphic to

$$\bigoplus_{i\in\mathbf{Z}} H_i(M_\bullet)[i],$$

the complex with $H_i(M_\bullet)$ in degree $i$ and all differentials zero.

(i) Let $k$ be a field. Show that every chain complex of $k$-modules is formal.

(ii) Give an example of a non-formal complex over a commutative ring A.

**3.** (i) Let A be a commutative ring and $I \subset A$ an ideal contained in the Jacobson radical of A. Show that the homomorphism $\mathcal{M}(A) \to \mathcal{M}(A/I)$, given by extension of scalars along the quotient map $\phi : A \to A/I$, is injective. Recall that $\mathcal{M}(A)$ denotes the monoid of isomorphism classes of f.g. projective A-modules. (Hint: Nakayama.)

(ii) Suppose that I is a nil ideal, i.e., that every element $x \in I$ is nilpotent. Let $\phi : A \to A/I$ be the quotient map. Show that the homomorphism $\phi^* : K_0(A) \to K_0(A/I)$ is invertible. (Hint: use the fact that idempotents lift along quotients by nil ideals in associative rings, and apply this to the ring of endomorphisms of $A^{\oplus n}$.)

**4.** Let $\phi : A \to B$ be a ring homomorphism which exhibits B as a f.g. free A-module of rank $d$. Then we have $[B] = d.[A] = d$ in $K_0(A)$. Show that the composites

$$K_0(A) \xrightarrow{\phi^*} K_0(B) \xrightarrow{\phi_*} K_0(A)$$

$$K_0(B) \xrightarrow{\phi_*} K_0(A) \xrightarrow{\phi^*} K_0(B)$$

are both given by multiplication by $d$.