Exercise sheet 7

1. (i) Let $F : \mathcal{A} \to \mathcal{A}'$ be an exact functor between abelian categories. Show that $\mathrm{Ker}(F) \subseteq \mathcal{A}$, the full subcategory spanned by objects A such that $F(A) \simeq 0$, is a Serre subcategory.

(ii) Let $F : \mathcal{A} \to \mathcal{A}'$ be an exact functor between abelian categories. Suppose that F admits a right adjoint G such that the co-unit transformation $FG \to \mathrm{id}$ is invertible (equivalently, G is fully faithful). Show that there is a canonical equivalence

$$\mathcal{A}/\,\mathrm{Ker}(F) \to \mathcal{A}'.$$

(iii) Let $\mathcal{A}$ be an abelian category and $\mathcal{B} \subseteq \mathcal{A}$ a Serre subcategory. Let $\mathcal{A}_0 \subseteq \mathcal{A}$ be a full subabelian subcategory such that if $A \in \mathcal{A}_0$ and $B \in \mathcal{B}$ is a subobject or quotient of A then also $B \in \mathcal{A}_0$. Show that the canonical functor

$$\mathcal{A}_0/(\mathcal{B} \cap \mathcal{A}_0) \to \mathcal{A}/\mathcal{B}$$

is fully faithful.

2. Let A be a ring and $f \in A$ an element.

(i) Let $\mathrm{Mod}_A(f^\infty) \subseteq \mathrm{Mod}_A$ denote the full subcategory of A-modules M that are f^∞-torsion (i.e., for every $x \in M$, $f^k x = 0$ for $k \gg 0$). Show that this is a Serre subcategory and that the canonical functor

$$\mathrm{Mod}_A/(\mathrm{Mod}_A(f^\infty)) \to \mathrm{Mod}_{A[f^{-1}]}$$

is an equivalence.

(ii) Assume that A is noetherian. Show that the canonical functor

$$\mathrm{Mod}^{\mathrm{fg}}_A/(\mathrm{Mod}^{\mathrm{fg}}_A(f^\infty)) \to \mathrm{Mod}^{\mathrm{fg}}_{A[f^{-1}]}$$

is fully faithful.

(iii) Let $B = A[f^{-1}]$. Show that every f.g. B-module N lifts to a f.g. A-module M such that $M \otimes_A B \simeq N$. Deduce that the canonical functor

$$\mathrm{Mod}^{\mathrm{fg}}_A/(\mathrm{Mod}^{\mathrm{fg}}_A(f^\infty)) \to \mathrm{Mod}^{\mathrm{fg}}_{A[f^{-1}]}$$

is an equivalence. (Hint: consider $N_{[A]} \in \mathrm{Mod}_A$, which may not be f.g. However you can find a surjection $A^{\oplus(I)} \to N_{[A]}$ from a free A-module indexed on a (possibly infinite) set I...)

3. Let A be a noetherian ring.

(i) Show that there is an injective homomorphism

$$G_0(A) \to G_0(A[T]).$$

(Hint: Note that ϕ admits a retraction in the category of commutative rings...)

(ii) If A is a field k, show that $\phi^* : \mathrm{G}_0(k) \to \mathrm{G}_0(k[\mathrm{T}])$ is an isomorphism.

4. Let A be an integral domain. Given an element $f \in \mathrm{A}$ and a point $p \in \big|\mathrm{Spec}(\mathrm{A})\big|$, the *value* of f at p, denoted $f(p)$, is the image of f by the homomorphism $\phi : \mathrm{A} \to \kappa(p)$. (Elements of A are thought of as "algebraic functions" on Spec(A).)

(i) Show that if an element f vanishes at the generic point η then $f = 0$.

(ii) Give an example to show that if A is not an integral domain, then an element $f \in \mathrm{A}$ can vanish at every point without being zero.

(Use the definition of $\big|\mathrm{Spec}(\mathrm{A})\big|$ given in the lecture, not the one using prime ideals.)