Exercise sheet 8

1. Let k be a field and $\mathrm{A} = k[\mathrm{T}_1, \dots, \mathrm{T}_n]$ the polynomial algebra on n generators.

(i) If k is algebraically closed, show that the closed points in $|\mathrm{Spec}(\mathrm{A})|$ are in bijection with tuples $(x_1, \dots, x_n) \in k^n$.

(ii) In general, let $\overline{k}$ be an algebraic closure of k and consider the automorphism group $\mathrm{G} = \mathrm{Aut}(\overline{k}/k)$. Show that there is a canonical action of G on the set of closed points of $|\mathrm{Spec}(\overline{k}[\mathrm{T}_1, \dots, \mathrm{T}_n])|$.

(iii) Show that the closed points in $|\mathrm{Spec}(\mathrm{A})|$ are in bijection with the G-orbits of the closed points of $|\mathrm{Spec}(\overline{k}[\mathrm{T}_1, \dots, \mathrm{T}_n])|$.

2. Let A be a noetherian local ring. Recall that $|\mathrm{Spec}(\mathrm{A})|$ has a unique closed point x.

(i) Show that $\mathrm{M} \in \mathrm{Mod}_{\mathrm{A}}^{\mathrm{fg}}$ is supported on $\mathrm{V}(\mathfrak{m}) \simeq \{x\}$ iff it is of finite length.

(ii) Show that the dévissage isomorphism $\mathrm{G}_0^{\{x\}}(\mathrm{A}) \simeq \mathbf{Z}$ sends $[\mathrm{M}] \mapsto \ell_{\mathrm{A}}(\mathrm{M})$, where $\ell_{\mathrm{A}}(\mathrm{M})$ denotes the length of M.

(iii) If A is regular, show that the intersection multiplicity is computed by the formula

$$\chi_{\mathrm{A}}(\mathrm{M}, \mathrm{N}) = \sum_i (-1)^i \, \ell_{\mathrm{A}}(\mathrm{Tor}_i^{\mathrm{A}}(\mathrm{M}, \mathrm{N}))$$

where M and N are A-modules with $\mathrm{Supp}_{\mathrm{A}}(\mathrm{M}) \cap \mathrm{Supp}_{\mathrm{A}}(\mathrm{N}) = \{x\}$ (x being the closed point of $|\mathrm{Spec}(\mathrm{A})|$).

3. Let k be an algebraically closed field and $\mathrm{A} = k[\mathrm{T}, \mathrm{U}]$. Let I and J be prime ideals of A defining *distinct* integral closed subsets $\mathrm{Y} = \mathrm{V}(\mathrm{I})$ and $\mathrm{Z} = \mathrm{V}(\mathrm{J})$ of codimension 1. Let p be a closed point of $|\mathrm{Spec}(\mathrm{A})|$ which lies in the intersection $\mathrm{Y} \cap \mathrm{Z}$, and let $\mathfrak{m}$ be the corresponding maximal ideal of A. Show that

$$\chi_{\mathrm{A}_\mathfrak{m}}(\mathrm{A}_\mathfrak{m}/\mathrm{I}\mathrm{A}_\mathfrak{m}, \mathrm{A}_\mathfrak{m}/\mathrm{J}\mathrm{A}_\mathfrak{m}) = \dim_k(\mathrm{A}_\mathfrak{m}/(\mathrm{I}+\mathrm{J})\mathrm{A}_\mathfrak{m}).$$

4. Let k be an algebraically closed field and $\mathrm{A} = k[\mathrm{T}_1, \mathrm{T}_2, \mathrm{T}_3, \mathrm{T}_4]$. Consider the ideals

$$\mathrm{I} = \langle \mathrm{T}_1, \mathrm{T}_2 \rangle \cap \langle \mathrm{T}_3, \mathrm{T}_4 \rangle = \langle \mathrm{T}_1\mathrm{T}_3, \mathrm{T}_1\mathrm{T}_4, \mathrm{T}_2\mathrm{T}_3, \mathrm{T}_2\mathrm{T}_4 \rangle$$
$$\mathrm{J} = \langle \mathrm{T}_1 - \mathrm{T}_3, \mathrm{T}_2 - \mathrm{T}_4 \rangle,$$

which define closed subsets $\mathrm{Y} = \mathrm{V}(\mathrm{I})$ and $\mathrm{Z} = \mathrm{V}(\mathrm{J})$ of $\mathrm{X} = |\mathrm{Spec}(\mathrm{A})|$.

(i) Show that Y has two irreducible components, each of codimension 2 in X.

(ii) Show that each component of Y intersects Z at exactly one closed point p in X.

(iii) Let $\mathfrak{m}$ be the maximal ideal of A corresponding to p. Compute the integers

$$\ell_{\mathrm{A}}(\mathrm{A}/(\mathrm{I}+\mathrm{J})), \qquad \ell_{\mathrm{A}_{\mathfrak{m}}}(\mathrm{A}_{\mathfrak{m}}/(\mathrm{I}+\mathrm{J})\mathrm{A}_{\mathfrak{m}}).$$

(iv) Compute the intersection number

$$\chi_{\mathrm{A}_{\mathfrak{m}}}(\mathrm{A}_{\mathfrak{m}}/\mathrm{I}\mathrm{A}_{\mathfrak{m}}, \mathrm{A}_{\mathfrak{m}}/\mathrm{J}\mathrm{A}_{\mathfrak{m}}).$$