Lecture 3

Algebraic K-theory and G-theory

3.1 Algebraic K-theory

Construction (Group completion):

$(X, \oplus)$ commutative monoid

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "\forall"'] & X^{gp} \arrow[d, dashed, "\exists!"] \\ & Y \text{ group} \end{tikzcd}$$

universal monoid homo. with X^{gp} a group (abelian)

Two constructions of X^{gp}:

1) formal differences: $\{(x,y) \mid x,y \in X\}/\sim$

$(x,y) \sim (x',y') \iff x \oplus y' \oplus z = x' \oplus y \oplus z$ for some $z \in X$

group operation on X^{gp}:

$(x,y) \oplus (x',y') = (x \oplus x', y \oplus y')$

2) linear combinations: $\mathbb{Z}[X]/\sim$

free abelian group w/ generators $x \in X$

and relations: $[x \oplus y] = [x] + [y]$

Ex: $(\mathbb{N}, +)^{gp} \cong \mathbb{Z}$

Exercise: Show 1) and 2) are canonically isomorphic. Show the universal property.

Construction: A = comm. ring

$\mathcal{M}(A)$ = monoid of iso. classes of objects of $\mathrm{Mod}_A^{\mathrm{fgproj}}$

operation: direct sum

$$K_0(A) := \mathcal{M}(A)^{gp}$$

Example: A local ring. (or PID)

$\Longrightarrow$ every $M \in \mathrm{Mod}_A^{\mathrm{fgproj}}$ is f.g. free

$\Longrightarrow$ $\mathcal{M}(A) \cong (\mathbb{N}, +)$ monoid isomorphism

$\Longrightarrow$ $K_0(A) \cong \mathbb{Z}$

Lemma: A = ring

(i) Every $x \in K_0(A)$ can be written as
$$[M] - [A^{\oplus n}], \quad \text{where } M \in \mathrm{Mod}_A^{\mathrm{fgproj}}, \ n \in \mathbb{N}$$

(ii) $M, N \in \mathrm{Mod}_A^{\mathrm{fgproj}}$. Then
$$[M] = [N] \in K_0(A)$$
$$\iff M \oplus A^{\oplus n} \cong N \oplus A^{\oplus n} \quad \text{for some } n \geq 0$$
(<u>stably equivalent</u>)

Proof: Recall every $M \in \mathrm{Mod}_A^{\mathrm{fgproj}}$ has
$$M \oplus M' \cong A^{\oplus n} \quad \text{for some } M' \in \mathrm{Mod}_A^{\mathrm{fgproj}}, \ n \geq 0.$$

(i) Clear that
$$x = [M] - [N]$$
for some $M, N \in \mathrm{Mod}_A^{\mathrm{fgproj}}$.

Write $N \oplus N' \cong A^{\oplus n}$ $(N' \in \mathrm{Mod}_A^{\mathrm{fgproj}})$.
$$\Rightarrow [N] = [A^{\oplus n}] - [N']$$
$$\Rightarrow x = [M] + [N'] - [A^{\oplus n}]$$
$$= [M \oplus N'] - [A^{\oplus n}].$$

(ii) $\quad [M] = [N]$ in $K_0(A)$

As formal differences:

$$(M, 0) \sim (N, 0)$$

$$\implies M \oplus 0 \oplus P \cong N \oplus 0 \oplus P \quad \text{for some } P \in \mathrm{Mod}_A^{\mathrm{fgproj}}$$

$$M \oplus P \cong N \oplus P$$

But $P \oplus P' \cong A^{\oplus n} \quad (\exists P', n \geq 0)$

$$\implies M \oplus A^{\oplus n} \cong N \oplus A^{\oplus n} \quad (\text{stably equiv.}) \qquad \blacksquare$$

<u>Remark</u>: Ring structure on $K_0(A)$:

$$[M] \cdot [N] := [M \otimes_A N]$$

unital, assoc., commutative.

3.2 G-theory

Def: A noetherian ring.

$G_0(A)$ = free abelian gp. generated by isomorphism classes of all f.g. modules with relations: for any

$$0 \to M' \to M \to M'' \to 0 \quad \text{short exact,}$$

$$[M] = [M'] + [M''] \quad \text{in } G_0(A).$$

Construction: $G_0(A)$ is a module over the comm. ring $K_0(A)$:

$$K_0(A) \otimes G_0(A) \longrightarrow G_0(A)$$
$$[P] \otimes [M] \longmapsto [P \otimes_A M]$$

Well-defined: if $0 \to M' \to M \to M'' \to 0$ exact then P flat, so $0 \to P \otimes_A M' \to P \otimes_A M \to P \otimes_A M'' \to 0$ exact

$$\Longrightarrow [P \otimes_A M] = [P \otimes_A M'] + [P \otimes_A M''].$$

<u>Remark</u>: $\exists$ $K_0(A) \longrightarrow G_0(A)$ canonical

$$[M] \longmapsto [M]$$

<u>Theorem</u>: If A is <u>regular</u>, then

$$K_0(A) \longrightarrow G_0(A)$$

is an isomorphism.

<u>Example</u>: If $A = k$ field, then

$$\mathrm{Mod}_k^{fg} = \mathrm{Mod}_k^{f.g.\,free}$$

$$\Rightarrow G_0(k) \cong K_0(k) \cong \mathbb{Z}.$$