Lecture 5

G-theory of coherent complexes

5.1. Truncations

Def: $M_\bullet \in Ch_A$, $n \in \mathbb{Z}$

$$\begin{tikzcd}
\tau_{\leq n}(M_\bullet) = (0 \arrow[r] & \operatorname{Coker}(d_{n+1}^{M_\bullet}) \arrow[r] & M_{n-1} \arrow[r, "d_{n-1}"] & M_{n-2} \arrow[r] & \cdots) \\
M_\bullet = (\cdots \arrow[r] \arrow[u] & M_n \arrow[r, "d_n"] \arrow[u] & M_{n-1} \arrow[r, "d_{n-1}"] \arrow[u, equal] & M_{n-2} \arrow[r] \arrow[u, equal] & \cdots)
\end{tikzcd}$$

Exercise: (i) $M_\bullet \longrightarrow \tau_{\leq n}(M_\bullet)$ induces isos.
$H_k(M_\bullet) \xrightarrow{\sim} H_k(\tau_{\leq n} M_\bullet)$ for all $k \leq n$.

(ii) $H_k(\tau_{\leq n} M_\bullet) = 0 \quad \forall k < n$

(iii) $M_\bullet \longrightarrow \tau_{\leq n}(M_\bullet)$ is a qis
iff $H_k(M_\bullet) = 0 \quad \forall k > n$.

Prop: There is an exact triangle
$$H_n(M)[n] \longrightarrow \tau_{\leq n}(M_\bullet) \longrightarrow \tau_{\leq n-1}(M_\bullet)$$
for every $M_\bullet \in Ch_A$, $\forall n \in \mathbb{Z}$.

5.2. G-theory of coherent complexes

Recall: There is a canonical homo.

$$G_0(A) \longrightarrow K_0(\mathrm{Coh}_A)$$
$$[M] \longmapsto [M[0]]$$

for any noetherian ring A.

Theorem: This homo. is an isomorphism.

Proof: $M_\bullet \in \mathrm{Coh}_A$ has homology groups concentrated in some range, say $[m, n]$.

$\Longrightarrow M_\bullet \underset{q\text{-}is}{\simeq} \tau_{\leq n}(M_\bullet)$ and $\tau_{\leq m-1}(M_\bullet) \underset{q\text{-}is}{\simeq} 0$

Exact triangles:

$$H_n(M_\bullet)[n] \to \tau_{\leq n}(M_\bullet) \to \tau_{\leq n-1}(M_\bullet)$$
$$H_{n-1}(M_\bullet)[n-1] \to \tau_{\leq n-1}(M_\bullet) \to \tau_{\leq n-2}(M_\bullet)$$
$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$
$$H_m(M_\bullet)[m] \to \tau_{\leq m}(M_\bullet) \to \tau_{\leq m-1}(M_\bullet)$$

$\Rightarrow$ relations in $K_0(\mathrm{Coh}_A)$:

$$[M_\bullet] = [\tau_{\leq n} M_\bullet] = [H_n(M_\bullet)[n]] + [\tau_{\leq n-1}(M_\bullet)] = \cdots$$
$$= \sum_{i=m}^{n} [H_i(M_\bullet)[i]] \qquad \text{(recursively)}$$

Recall $[H_i(M_\bullet)[i]] = (-1)^i \cdot [H_i(M_\bullet)]$.

$$\Longrightarrow \quad [M_\bullet] = \sum_{i \in \mathbb{Z}} (-1)^i \cdot [H_i(M_\bullet)] \qquad \forall M_\bullet \in \mathrm{Coh}_A .$$

Now define an inverse map

$$\begin{aligned} K_0(\mathrm{Coh}_A) &\longrightarrow G_0(A) \\ [M_\bullet] &\longmapsto \sum_{i \in \mathbb{Z}} (-1)^i [H_i(M_\bullet)] \end{aligned}$$

The formula above implies that this is inverse to the map $G_0(A) \to K_0(\mathrm{Coh}_A)$ in question. ■

5.3. K-theory vs. G-theory

Theorem: If A is regular, then the canonical homo. $K_0(A) \longrightarrow G_0(A)$ is an isomorphism.

Proof:

$$
\begin{tikzcd}
K_0(A) \arrow[r] \arrow[d, "\wr"'] \arrow[dr, phantom, "\circ"] & G_0(A) \arrow[d, "\wr"] \\
K_0(\mathrm{Perf}_A) \arrow[r] & K_0(\mathrm{Coh}_A)
\end{tikzcd}
$$

commutative square
vertical arrows
are isos

If A regular, then $G_0(A) \longrightarrow K_0(\mathrm{Coh}_A)$
$[M] \longmapsto [M[0]]$
factors through $K_0(\mathrm{Perf}_A)$.

$$
\begin{tikzcd}
K_0(A) \arrow[r] \arrow[d, "\wr"'] & G_0(A) \arrow[d, "\wr"] \arrow[dl] \\
K_0(\mathrm{Perf}_A) \arrow[r] \arrow[u, phantom, shift left=6, "\circ"] & K_0(\mathrm{Coh}_A) \arrow[u, phantom, shift right=6, "\circ"]
\end{tikzcd}
$$

Both triangles commute.
$\Rightarrow$ $G_0(A) \rightarrow K_0(\mathrm{Perf}_A)$ has inverses on both sides.
$\Rightarrow$ all maps in the diagram are isos. ■

5.4. Tor-amplitude of complexes.

<u>Theorem</u>: A : noetherian, $M_\bullet \in \mathrm{Coh}_A$.
Then $M_\bullet \in \mathrm{Perf}_A \iff M_\bullet$ of finite Tor-amplitude

<u>Definition</u>: A: ring, $M_\bullet \in \mathrm{Ch}_A$, $a \leq b \in \mathbb{Z}$
$M_\bullet$ is <u>of Tor-amplitude $[a,b]$</u>

$$\iff H_i(M_\bullet \overset{L}{\underset{A}{\otimes}} N) = 0 \quad \forall N \in \mathrm{Mod}_A, \ \forall i \notin [a,b]$$

$M_\bullet$ is <u>of finite Tor-amplitude</u> if it is of Tor-amplitude $[a,b]$ for some $a \leq b \in \mathbb{Z}$.

<u>Remark</u>: Can compute $M_\bullet \overset{L}{\underset{A}{\otimes}} N$ by using any $P_\bullet \xrightarrow[\text{qis}]{} M_\bullet$ where P_i are projective (but not necessarily f.g., and $P_\bullet$ not necessarily finite).

Alternatively, can resolve N. This involves double complexes and totalization to define the tensor product of chain complexes.
<u>Reference</u>: [Gelfand-Manin, III, §7.8; III, Ex. 6].

__Remark:__ If $M_\bullet$ is of Tor-amplitude $[a,b]$, then $H_i(M_\bullet \otimes_A N) = 0$ $\forall N \in \mathrm{Mod}_A$, $\forall i \notin [a,b]$.
Taking $N = A$, we have in particular:

$$H_i(M_\bullet) = 0 \qquad \forall\, i \notin [a,b].$$

__Lemma:__ $M_\bullet \in \mathrm{Ch}_A$ bounded below ($M_i = 0$ $\forall i \ll 0$)
M_i flat $\forall i$
$M_\bullet$ of Tor-amplitude $[a,b]$

Then: $\mathrm{Coker}(d_{b+1})$ is flat.
In particular: $\tau_{\leq b}(M_\bullet)$ is a complex of flat modules.

__Proof:__ Sufficient to show: $\mathrm{Tor}_1^A(\mathrm{Coker}(d_{b+1}), N) = 0$ $\forall N \in \mathrm{Mod}_A$

Resolution of $\mathrm{Coker}(d_{b+1})$:

$$\begin{tikzcd}
\cdots \arrow[r] & M_{b+2} \arrow[r] & M_{b+1} \arrow[r, "d_{b+1}"] \arrow[d] & M_b \arrow[r] \arrow[d] & 0 & P_\bullet \arrow[d, "\mathrm{qis}"'] \\
 & & 0 \arrow[r] & \mathrm{Coker}(d_{b+1}) \arrow[r] & 0 & \mathrm{Coker}(d_{b+1})[0]
\end{tikzcd}$$

where the upper row $P_\bullet$ is a complex of flat modules.
In particular, $P_i \otimes_A (-)$ is exact for all i.

$\Rightarrow$ $M_{b+2} \otimes_A N \to M_{b+1} \otimes_A N \to M_b \otimes_A N$ exact $\forall N$

$\Rightarrow$ $\mathrm{Tor}_1^A(\mathrm{Coker}(d_{b+1}), N) \cong H_1(P_\bullet \otimes_A N) = 0$. ■

Corollary: $M_\bullet \in Ch_A$. Then $M_\bullet$ of Tor-amplitude $[a,b]$ iff $M_\bullet \underset{qis}{\simeq} P_\bullet$ where P_i are flat, $P_i = 0 \ \forall i \notin [a,b]$.

Proof: One direction is clear.
If $M_\bullet$ is of Tor-amplitude $[a,b]$, then choose $Q_\bullet \underset{qis}{\longrightarrow} M_\bullet$ a projective resolution (possibly infinite). This can be chosen such that $Q_i = 0 \ \forall i < a$.

$P_\bullet := \tau_{\leq b}(Q_\bullet)$ is a complex of flat modules (by the Lemma).

$Q_\bullet \xrightarrow{qis} \tau_{\leq b}(Q_\bullet) = P_\bullet$ qis since $H_k(Q_\bullet) = 0$
$= H_k(M_\bullet) = 0 \ \forall k < b$

$\rightsquigarrow M_\bullet \underset{qis}{\longleftarrow} Q_\bullet \underset{qis}{\longrightarrow} P_\bullet$ ■

Proof of Theorem:

If $M_\bullet$ is perfect $\Longrightarrow \exists\, P_\bullet \xrightarrow{q.i.s} M_\bullet$, $P_\bullet$ finite fgproj

$H_i(M_\bullet \overset{L}{\underset{A}{\otimes}} N) = H_i(P_\bullet \underset{A}{\otimes} N)$ is bounded by the length of $P_\bullet$

If $M_\bullet \in \mathrm{Coh}_A$ of Tor-amplitude $[a,b]$

$\Rightarrow M_\bullet \underset{q.i.}{\simeq} P_\bullet$ where P_i flat, $P_i = 0 \;\forall i \notin [a,b]$

Induction on the length of $P_\bullet$ $(b-a)$

$\underline{b=a}$: $P_\bullet \simeq P_i[i]$.

$M_\bullet$ coherent $\Rightarrow P_\bullet$ coherent $\Rightarrow P_i$ f.g. flat

$\Rightarrow P_i$ f.g. proj.

(A noetherian) ■

$\underline{b>a}$: We have an exact triangle

$P_b[b-1] \to \sigma_{\leq b-1}(P_\bullet) \to P_\bullet$

where $\sigma_{\leq b-1}(P_\bullet)$ is the "brutal" truncation

$(0 \to P_{b-1} \to P_{b-2} \to \cdots \to P_a \to 0)$.

The morphism $P_b[b-1] \to \sigma_{\leq b-1}(P_\bullet)$ is induced by $P_b \xrightarrow{d_b} P_{b-1}$ and its cone is iso. to $P_\bullet$.

By induction, $\sigma_{\leq b-1}(P_\bullet)$ is perfect and $P_b[b-1]$ is perfect. Hence the cone is perfect. ■