Motivic homotopy theory in derived algebraic geometry

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Preface

1. What is motivic homotopy theory?

1.1. The existence of a *motivic cohomology theory* was first conjectured by A. Beilinson [Bei87]. This cohomology theory was expected to be universal with respect to mixed Weil cohomologies like ℓ-adic cohomology or algebraic de Rham cohomology; that is, there should be cycle class maps from the rational motivic cohomology groups to, say, ℓ-adic cohomology.

Further, this cohomology theory was expected to arise from a category of coefficients, the category of so-called *motivic complexes*. Hence for each scheme S, Beilinson conjectured the existence of categories $\mathcal{DM}(\mathrm{S})$, admitting Grothendieck's formalism of six operations: there should be operations $(f^*, f_*, f_!, f^!, \otimes, \underline{\mathrm{Hom}})$ acting on these categories, for $f : \mathrm{T} \to \mathrm{S}$ a morphism of schemes, satisfying various compatibilities like base change and projection formulas.

After the work of many mathematicians including J. Ayoub, D.-C. Cisinski, F. Déglise, M. Levine, and V. Voevodsky, we are close to the theory envisioned by Beilinson: we have categories of motivic complexes over general base schemes, satisfying the formalism of six operations at least partially. We refer the reader to [CD09] for the state of the art.

1.2. In algebraic topology, the analogue of motivic complexes are complexes of abelian groups: the derived category $\mathcal{D}(\mathcal{A}b)$ is the category of coefficients for singular cohomology. It turns out that $\mathcal{D}(\mathcal{A}b)$ is the linear version of another category $\mathcal{S}pt$ (whose objects are called spectra), in the sense that it can be identified with the category of modules over a spectrum $\mathbf{Z}_{\mathcal{S}pt}$ in $\mathcal{S}pt$. In the category $\mathcal{S}pt$, we have representability not only of singular cohomology, but of generalized cohomology theories like complex K-theory and complex cobordism.

We can form an analogue of the category $\mathcal{S}pt$ in algebraic geometry. F. Morel, and Voevodsky constructed for each scheme S, categories $\mathcal{SH}(\mathrm{S})$ of *motivic spectra*. The thesis of Ayoub [Ayo07] established the construction of the six operations on motivic spectra, following ideas of Voevodsky.

Just as in topology, the category of modules over the spectrum $\mathbf{Z}_{\mathrm{S}}$, representing motivic cohomology, is equivalent to the category $\mathcal{DM}(\mathrm{S})$ (at least when S is the spectrum of a field of characteristic zero; see [RØ08]). In the category $\mathcal{SH}(\mathrm{S})$ we have representability of generalized motivic cohomology theories like homotopy invariant algebraic K-theory and algebraic cobordism. Mixed Weil cohomologies are also representable by motivic spectra (see [CD12]).

When a cohomology theory is representable as a motivic spectrum, we may view cohomology classes as morphisms in the category $\mathcal{SH}(\mathrm{S})$. This gives us the possibility to take advantage of the power of the formalism of six operations, of which only shadows can be seen at the level of cohomology groups (Künneth formulas, Gysin maps, etc.).

2. Why derived schemes?

2.1. As was expected by Beilinson, motivic cohomology is closely related to intersection theory and algebraic cycles. For example, Levine proved that over a field, it can be computed as the higher Chow groups defined by S. Bloch (see [Lev06]).

It is not currently known how to extend this comparison to more general bases, as conjectured by Beilinson; Levine's proof uses a highly technical moving lemma which is only known for fields. This is related to the lack of a good intersection theory over general base schemes (say regular and of finite type over $\mathrm{Spec}(\mathbf{Z})$), a problem posed by A. Grothendieck in [SGA 6] and studied indirectly by D. Quillen and R. Thomason in their work on algebraic K-theory.

We will not have anything to say about this problem in this thesis, but it was the motivation for us to pass to the world of *derived algebraic geometry*.

2.2. Derived algebraic geometry is an extension of classical algebraic geometry where we allow schemes to have "higher nilpotents".

For example, let Z and Z' be closed subschemes of a regular scheme X, defined by ideals $\mathcal{I}$ and $\mathcal{I}'$, respectively. Let us assume that Z and Z' intersect *properly*, so that $\mathrm{codim}(Z \cap Z', X) = \mathrm{codim}(Z, X) + \mathrm{codim}(Z', X)$. It is well-known that the scheme-theoretic intersection $Z \times_X Z'$ does not capture the correct intersection multiplicity in the non-transverse case. Rather, according to Serre's intersection formula (see [Ser00]), we must take all the groups

$$\mathrm{Tor}_i^{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/\mathcal{I}_x, \mathcal{O}_{X,x}/\mathcal{I}'_x) \qquad (i \geqslant 0)$$

into account, for each generic point x of $Z \times_X Z'$.

On the other hand, we can take the intersection in the world of derived algebraic geometry, i.e. the derived fibred product $Z \times^{\mathbf{R}}_X Z'$. Its underlying classical scheme coincides with $Z \times_X Z'$, but when Z and Z' do not intersect transversally, it has higher nilpotents. More precisely, its structure sheaf is a sheaf of *simplicial commutative rings*; we recover the sheaf $\mathcal{O}_{Z \times_X Z'}$ as $\pi_0(\mathcal{O}_{Z \times^{\mathbf{R}}_X Z'})$, but it also has higher homotopy groups

$$\pi_i(\mathcal{O}_{Z \times^{\mathbf{R}}_X Z'}) \qquad (i \geqslant 0)$$

whose stalks are none other than the Tor groups written above. The data contained in these higher homotopy groups is what we think of as "higher" nilpotents.

This observation is what suggested to us that derived algebraic geometry could be a more natural setting for the study of motives.

3. What we do in this text

In this thesis we will show that the motivic homotopy categories $\mathcal{SH}(S)$ extend to the world of derived algebraic geometry, as well as the full formalism of six operations.

In Chapter 0, we review some preliminaries about theories of $(\infty, 1)$-categories and derived algebraic geometry. The reader who is already familiar with these theories may want to skim this chapter in order to acquaint himself with the notation we use.

In Chapter 1, we construct the unstable and stable motivic homotopy categories over a derived scheme. We prove our first main result, the analogue of Morel–Voevodsky's localization theorem for motivic spaces.

In Chapter 2, we construct the formalism of six operations on the stable motivic homotopy category. We use the formalism of $(\infty, 2)$-categories of correspondences introduced in [GR16].

4. Relation with previous work

4.1. Motivic homotopy theory over classical base schemes was introduced in [MV99], using the language of model categories. An $(\infty, 1)$-categorical construction was given in [Rob14]. Our definition is essentially a straightforward generalization to the setting of derived schemes. The only subtlety is that, unlike Morel–Voevodsky, we do not impose any finiteness conditions on our base schemes. This means that descent (by which we mean descent with respect to Čech covers) is no longer equivalent to hyperdescent (by which we mean descent with respect to arbitrary hypercovers). Here we work with the weaker notion of descent.

The main result of Chapter 1, the localization theorem, was proved by Morel–Voevodsky in *loc. cit.* Our proof follows the same general strategy, but since we impose a weaker descent condition on our sheaves, we have to work a bit more.

4.2. The formalism of six operations in motivic homotopy theory, in the setting of classical schemes, has been constructed by J. Ayoub in his thesis [Ayo07], following ideas of Voevodsky. We use part of his work as input in our construction.

However, Ayoub works with the language of triangulated categories, and so does not obtain the full homotopy-coherent system of compatibilities between the various operations. An $(\infty, 1)$-categorical lift of the six functor formalism was completed in the thesis of Robalo [Rob14], using the framework of [LZ12] based on multi-simplicial sets. In this work, we follow a different approach, developed by Gaitsgory–Rozenblyum in [GR16], based on $(\infty, 2)$-categories of correspondences. The relation between the two is explained in *loc. cit.* However, we slightly modify the approach of Gaitsgory–Rozenblyum in order to encode projection formulas; their method of encoding projection formulas cannot be applied to the category of motivic spectra, because we do not know if the canonical functors $\mathcal{SH}(\mathrm{X}) \otimes_{\mathcal{SH}(\mathrm{S})} \mathcal{SH}(\mathrm{Y}) \to \mathcal{SH}(\mathrm{X} \times_{\mathrm{S}} \mathrm{Y})$ are equivalences.

5. What is not covered in this text

5.1. Using the framework we set up in this text, it is possible to give a definition of motivic cohomology, homotopy invariant algebraic K-theory, and algebraic cobordism of derived schemes. One can also prove that these cohomology theories are insensitive to higher nilpotent thickenings (just as they are known to be insensitive to usual nilpotent thickenings).

In a sequel to this work we will apply this to construct virtual fundamental classes in oriented generalized motivic Borel–Moore homology theories.

5.2. In this text we work with the version of derived algebraic geometry where commutative rings are replaced by *simplicial commutative rings*. It is possible to consider other contexts for derived algebraic geometry, like spectral algebraic geometry, where one uses *connective* $\mathbf{E}_\infty$*-ring spectra*. Another possibility, lying between simplicial commutative rings and connective $\mathbf{E}_\infty$-ring spectra, is *connective* $\mathbf{E}_\infty$*-dg-algebras* over a commutative ring.

The results of Chapter 1 are true in all of these contexts, and the proofs we have provided generalize *mutatis mutandis*; see [Kha16]. There is some subtlety however: there are two possible notions of smoothness in spectral algebraic geometry, corresponding to two possible choices for the affine line. Spectral smoothness, the version of smoothness based on the cotangent complex, corresponds to the spectral affine line. On the other hand, classical smoothness, by which we mean flat and smooth on underlying classical schemes, corresponds to the classical affine line. The correct definition of the motivic homotopy category turns out to be the one using the spectrally smooth site and the spectral affine line. However, a surprising observation is that, over a classical scheme, the motivic homotopy category may not coincide with the Morel–Voevodsky motivic homotopy category; for example, over any field of positive characteristic, the spectrally smooth site is different from the usual smooth site.

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CHAPTER 0

Preliminaries

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1. Introduction

In this chapter we give a very brief survey of the theories of $(\infty,1)$-categories and derived algebraic geometry.

1.1. $(\infty,1)$-categories. Let R be a commutative ring. Recall that for R-modules M and N, the groups $\mathrm{Tor}_i^{\mathrm{R}}(\mathrm{M},\mathrm{N})$ are shadows of a chain complex $\mathrm{M} \otimes_{\mathrm{R}}^{\mathbf{L}} \mathrm{N}$ which is well-defined only up to *quasi-isomorphism*: it depends on a choice of projective resolution. In other words, the abelian category of chain complexes of R-modules is too rigid for the purposes of homological algebra; categorical constructions like limits and colimits will not behave well with respect to quasi-isomorphism. Instead, the language of homotopy theory is appropriate.

An abstract homotopy theory is the datum of a category $\mathbf{C}$ together with a class $\mathbf{W}$ of morphisms called *weak equivalences*. Such a pair $(\mathbf{C},\mathbf{W})$ is a presentation of an $(\infty,1)$-category. For example, the $(\infty,1)$-category of homotopy types or *spaces* is presented equivalently by topological spaces (up to weak homotopy equivalence) or simplicial sets (also up to weak homotopy equivalence), among many other possibilities.

In fact, the collection of homotopy theories is itself a homotopy theory, and this homotopy theory is equivalent to the homotopy theory of $(\infty,1)$-categories[1]. In particular, the homotopy theory of homotopy theories can be taken as a model for $(\infty,1)$-categories. However, this approach can be problematic. After all, a pair $(\mathbf{C},\mathbf{W})$ is itself only well-defined up to weak equivalence of homotopy theories[2], which is much weaker than equivalence of the underlying category $\mathbf{C}$, and it is not clear how to extract $(\infty,1)$-categorical information like (co)limits from it, as they cannot just be computed as (co)limits in the category $\mathbf{C}$. Even more simply, morphisms in the associated $(\infty,1)$-category are not just morphisms in the category $\mathbf{C}$.

Classically, these problems have been solved by using D. Quillen's theory of *model categories*. A model structure on a pair $(\mathbf{C},\mathbf{W})$ is the data of classes of cofibrations and fibrations subject to various axioms. The purpose of this structure is to allow computation of $(\infty,1)$-categorical (co)limits as usual categorical (co)limits in the category $\mathbf{C}$, after suitable (co)fibrant replacement of the diagram. For example, there are projective and injective model structures on the category of chain complexes, where cofibrant or fibrant replacement corresponds to choosing projective or injective resolutions, respectively. Similarly, sets of morphisms in the $(\infty,1)$-category can be computed as the set of morphisms in the category $\mathbf{C}$, after taking cofibrant and fibrant replacements of the objects.

Though this approach is very powerful, the choice of a specific model-categorical presentation adds a factor of arbitrariness to all constructions and proofs, and requires some additional effort in checking that all constructions performed are in fact the homotopically correct ones. This effort can be nontrivial.[3]

The modern approach consists in forgetting about presentations entirely, and instead choosing a more transparent model for $(\infty,1)$-categories, like the *quasi-categories* of A. Joyal or *complete Segal spaces* of C. Rezk. For example, a quasi-category is by definition a simplicial set satisfying the weak Kan condition; an object of a quasi-category is a 0-simplex, a morphism is a

[1] This can be made into a mathematically rigourous statement after the work of [BK12].

[2] Weak equivalence of homotopy theories *is* precisely equivalence of $(\infty,1)$-categories.

[3] The theory gets very technical very quickly. For example, let $\mathbf{C}$ be a small $(\infty,1)$-category. The $(\infty,1)$-category of presheaves on $\mathbf{C}$ can be presented by the injective model structure on the category of simplicially enriched functors between the simplicially enriched category associated to $\mathbf{C}^{\mathrm{op}}$, and the simplicially enriched category of simplicial sets. Compare with Footnote 4.

1-simplex, and a functor between quasi-categories is a morphism of simpicial sets.[4] Similarly, (co)limits and other categorical constructions have intrinsic quasi-categorical descriptions.

Let us consider a simple example. Suppose for instance that we have a diagram

$$\begin{tikzcd} & \mathrm{X}_0 \arrow[d] \\ \mathrm{X}_1 \arrow[r] & \mathrm{X} \end{tikzcd}$$

in the $(\infty, 1)$-category of spaces.

The classical way to compute its limit would be choose a presentation of the $(\infty, 1)$-category of spaces, for example by the model category of topological spaces, and to compute the homotopy limit in this model structure. Hence one would first replace this diagram by a diagram in the category of topological spaces, where each of the objects is fibrant (in the chosen model structure), and at least one of the morphisms $\mathrm{X}_i \to \mathrm{X}$ is a fibration (in the chosen model structure). A topological space with the homotopy type of the homotopy limit would then be given by the usual categorical limit of the resulting diagram.

In the $(\infty, 1)$-categorical approach, the limit of the diagram is simply the space of pairs (x_0, x_1, α), where x_i is a point of X_i and α is a path in the space X between the images of the points x_0 and x_1. More precisely, each x_i is a morphism $\mathrm{pt} \to \mathrm{X}_i$, where pt denotes the terminal object in the $(\infty, 1)$-category of spaces, and α is a commutative square

$$\begin{tikzcd} \mathrm{pt} \arrow[r, "x_0"] \arrow[d, "x_1"] & \mathrm{X}_0 \arrow[d] \\ \mathrm{X}_1 \arrow[r] & \mathrm{X}. \end{tikzcd}$$

This is nothing more than the universal property of the $(\infty, 1)$-categorical limit.

1.2. Derived algebraic geometry. The $(\infty, 1)$-category of simplicial commutative rings is an enlargement of the ordinary category of commutative rings where we have objects like derived tensor products $\mathrm{A} \otimes^{\mathbf{L}}_{\mathrm{R}} \mathrm{B}$, for a commutative ring R and R-algebras A and B.

By carefully replacing commutative rings by simplicial commutative rings in the definition of *scheme* (and hence passing to the world of $(\infty, 1)$-categories), we obtain the definition of *derived scheme*. Any derived scheme S has an underlying classical scheme S_{cl}, and the relationship between S and S_{cl} is analogous to the relationship between the classical scheme S_{cl} and its underlying reduced scheme $\mathrm{S}_{\mathrm{cl,red}}$.

In the same way that natural constructions involving algebraic varieties force one into the world of schemes, derived schemes also arise naturally from considerations involving their classical counterparts.

For example, let $\mathrm{S} = \mathrm{Spec}(\mathrm{R})$ be an affine scheme and let $\mathrm{X} = \mathrm{Spec}(\mathrm{A})$ and $\mathrm{Y} = \mathrm{Spec}(\mathrm{B})$ be affine schemes over S. The *derived fibred product*

$$\mathrm{X} \overset{\mathbf{R}}{\underset{\mathrm{S}}{\times}} \mathrm{Y} = \mathrm{Spec}(\mathrm{A} \overset{\mathbf{L}}{\underset{\mathrm{R}}{\otimes}} \mathrm{B})$$

is an important example of a derived scheme. When X or Y is flat over S, this coincides with the classical fibred product $\mathrm{X} \times_{\mathrm{S}} \mathrm{Y} = \mathrm{Spec}(\mathrm{A} \otimes_{\mathrm{R}} \mathrm{B})$. In general, the derived tensor product contains information that cannot be recovered from the ordinary tensor product (cf. Serre's intersection formula, which in certain situations computes intersection multiplicities in terms of the groups $\mathrm{Tor}^{\mathrm{R}}_i(\mathrm{A}, \mathrm{B})$).

[4] The quasi-category of presheaves on a small quasi-category $\mathbf{C}$ is the internal hom $\underline{\mathrm{Hom}}_{\mathrm{sSets}}(\mathbf{C}^{\mathrm{op}}, \mathcal{S}pc)$, where $\mathcal{S}pc$ is the quasi-category of spaces. Compare with Footnote 3.

For another example[5], let $\mathrm{S} = \mathrm{Spec}(\mathrm{R})$ be an affine scheme and $f_0, \ldots, f_n \in \mathrm{R}$ a set of global sections. The closed derived subscheme cut out by the equations f_i is defined as

$$\mathrm{Z} = \mathrm{Spec}(\mathrm{R} \overset{\mathbf{L}}{/} (f_i)_i),$$

where we have written $\mathrm{R}/^{\mathbf{L}}(f_i)_i$ for the Koszul complex of the sequence $(f_i)_i$, which can be defined as a simplicial commutative ring. When $(f_i)_i$ is a *regular sequence*, the Koszul complex is quasi-isomorphic to the usual quotient $\mathrm{R}/(f_i)_i$, so Z is identical to the classical scheme $\mathrm{Spec}(\mathrm{R}/(f_i)_i)$, which is in this case a regularly embedded or lci closed subscheme of S. In general, Z is a derived scheme with underlying classical scheme

$$\mathrm{Z}_{\mathrm{cl}} = \mathrm{Spec}(\mathrm{R}/(f_i)_i).$$

These examples can both be globalized to non-affine base schemes S. To some extent, they explain the ubiquity of flatness and regularity/lci assumptions in algebraic geometry: these are precisely the conditions that guarantee that the relevant derived schemes are actually classical.

1.3. Conventions. We will use the language of $(\infty, 1)$-categories freely throughout the text. Though we will use the language in a model-independent way, we fix for concreteness the model of quasi-categories as developed by A. Joyal and J. Lurie.

For simplicity we adopt the following conventions:

The term "category" means, by default, "$(\infty, 1)$-category" (= quasi-category). When we want to refer to an ordinary category, we will use the term "ordinary category" or "$(1, 1)$-category".

We will say that a morphism in an $(\infty, 1)$-category is *invertible* or an *isomorphism* (as in [Joy04] and [GR16]) where some authors might use the word *equivalence* (e.g. [Lur09b]). We will use the symbol "=" for isomorphic objects in an $(\infty, 1)$-category, as the notion of equality simply does not exist.

The term "2-category" means, by default, "$(\infty, 2)$-category". When we want to refer to an ordinary 2-category, we will use the term "$(2, 2)$-category" or "ordinary 2-category". For us, this term always means *weak* 2-category (a.k.a. *bicategory*). An ordinary 2-category in which the 2-morphisms are invertible will be called a "$(2, 1)$-category".

The term "scheme" means, by default, "derived scheme". When we want to refer to a classical scheme, we will use the term "classical scheme".

Our focus is this chapter is on giving statements. We only attempt to give rigourous proofs when we do not know a reference in the literature. In particular, in this chapter, we will "define" $(\infty, 1)$-categories and $(\infty, 1)$-functors only on objects and 1-morphisms; the reader will find their precise constructions in the literature.

1.4. Organization of this chapter. This chapter is a very brief survey of some parts of the theories of $(\infty, 1)$-categories and of derived algebraic geometry that we will use in this text.

In Sect. 2 we briefly review several notions from the theory of $(\infty, 1)$-categories that will play an important role in this text: stable $(\infty, 1)$-categories, presheaves, filtered and sifted colimits, arenas (more commonly known as presentable $(\infty, 1)$-categories), toposes, and sites. References for this section are [Lur09b], [Lur16], and [Joy04].

In Sect. 3 we review some $(\infty, 2)$-category theory, and define an $(\infty, 2)$-category **Arenamod** which will play an important role in Chapter 2. We follow the treatment of [GR16].

[5]The reader will observe that this is in fact a special case of the first example.

In Sects. 4–6 we give a very brief exposition of the theory of derived schemes. We follow the functorial approach of Toën–Vezzosi, see [TV08] and [MT10].

2. $(\infty, 1)$-CATEGORIES

References for this section are [Lur09b], [Lur16], and [Joy04].

2.1. $(\infty, 1)$-categories.

2.1.1. In the world of $(\infty, 1)$-categories, all sets are replaced by *spaces*. Hence instead of elements of sets, we have points in spaces; instead of equality between two elements, we have paths between points, and so on.

For example, in any $(\infty, 1)$-category $\mathbf{C}$, we have for any two objects x and y a *mapping space*

$$\mathrm{Maps}_{\mathbf{C}}(x, y) \in \mathcal{S}pc$$

in the $(\infty, 1)$-category of spaces.

2.1.2. Any ordinary category can be viewed as an $(\infty, 1)$-category, with the same objects and discrete mapping spaces. This defines a fully faithful functor from the $(\infty, 1)$-category of ordinary categories to the $(\infty, 1)$-category of $(\infty, 1)$-categories.

Conversely, given an $(\infty, 1)$-category $\mathbf{C}$, we can form its underlying ordinary category, denoted $(\mathbf{C})^{\mathrm{ordn}}$, with the same objects and with morphisms given by connected components of mapping spaces:

$$\mathrm{Hom}_{(\mathbf{C})^{\mathrm{ordn}}}(x, y) = \pi_0(\mathrm{Maps}_{\mathbf{C}}(x, y)) \qquad (x, y \in \mathbf{C}).$$

The assignment $\mathbf{C} \mapsto (\mathbf{C})^{\mathrm{ordn}}$ defines a left adjoint to the above inclusion.

2.1.3. In the sequel, the term "category" will mean "$(\infty, 1)$-category" by default; when we need to refer to the classical notion of category, we will say "ordinary category".

2.1.4. We now recall some standard features of ordinary category theory which have analogues in this world.

2.1.5. Given a category $\mathbf{C}$, there are objects and morphisms. Morphisms can be *invertible*, or *isomorphisms*. There is a category $\mathcal{A}rrows(\mathbf{C})$ whose objects are morphisms in $\mathbf{C}$, and morphisms are commutative squares.

2.1.6. A category $\mathbf{C}$ has an underlying space, denoted $(\mathbf{C})^{\mathcal{S}pc}$.

The category of spaces embeds fully faithfully into the category of categories, and the assignment $\mathbf{C} \mapsto (\mathbf{C})^{\mathcal{S}pc}$ is right adjoint to the inclusion. We also use the term ∞-*groupoid* for a category in the essential image of this embedding.

2.1.7. Given a category $\mathbf{C}$, there is an opposite category $(\mathbf{C})^{\mathrm{op}}$ obtained by reversing the directions of morphisms.

2.1.8. Given two categories $\mathbf{C}$ and $\mathbf{D}$, we have a category $\mathcal{F}unct(\mathbf{C}, \mathbf{D})$ whose objects are functors $u : \mathbf{C} \to \mathbf{D}$. There are notions of *fully faithful*, *essentially surjective*, or *equivalence*, for any such functor. A functor is *exact* (resp. *right-exact*, *left-exact*) if it commutes with finite colimits and limits (resp. with finite colimits, resp. with finite limits).

2.1.9. Given a category $\mathbf{C}$, there is a notion of full subcategory of $\mathbf{C}$.

2.1.10. Given two categories $\mathbf{C}$ and $\mathbf{D}$, there is a notion of adjunction $(u : \mathbf{C} \to \mathbf{D}, v : \mathbf{D} \to \mathbf{C})$.

Given an adjunction as above and a morphism $f : u(c) \to d$ with c an object of $\mathbf{C}$ and d an object of $\mathbf{D}$, we say that the corresponding morphism $g : c \to v(d)$ is obtained by *right transposition* from f. Conversely, f is obtained by *left transposition* from g.

2.1.11. Given a category $\mathbf{C}$ and an object c, there are slice categories $\mathbf{C}_{/c}$ and ${}_{c\backslash}\mathbf{C}$. More generally, given a functor $u : \mathbf{I} \to \mathbf{C}$, there are slice categories $\mathbf{C}_{/u}$ and ${}_{u\backslash}\mathbf{C}$.

Also, given functors $u : \mathbf{C}_1 \to \mathbf{C}$ and $v : \mathbf{C}_2 \to \mathbf{C}$, there is a comma category $(u \downarrow v)$ whose objects are triples (c_1, c_2, f), with f a morphism $u(c_1) \to v(c_2)$ in $\mathbf{C}$.

Taking $\mathbf{C}_1$ (resp. $\mathbf{C}_2$) to be the trivial category, the functor u (resp. v) defines an object $c \in \mathbf{C}$, and we write the comma category $(u \downarrow v)$ as $(c\backslash v)$ (resp. (u/c)).

Note that when we further take v (resp. u) to be the identity functor of $\mathbf{C}$, we recover the slice categories $(c\backslash \mathrm{id}_{\mathbf{C}}) = {}_{c\backslash}\mathbf{C}$ (resp. $(\mathrm{id}_{\mathbf{C}}/c) = \mathbf{C}_{/c}$).

2.1.12. Let $\mathbf{I}$ be a category. Given an $\mathbf{I}$-indexed diagram $(c_i)_{i\in\mathbf{I}}$ in a category $\mathbf{C}$, i.e. a functor $(i \in \mathbf{I}) \mapsto (c_i \in \mathbf{C})$, there is a notion of colimit (resp. limit), denoted $\varinjlim_{i\in\mathbf{I}}$ (resp. $\varprojlim_{i\in\mathbf{I}}$), satisfying the expected universal property.

2.1.13. The collection of (small) categories forms a (large) category $(\infty, 1)$-$\mathcal{C}at$.

2.1.14. Given a category $\mathbf{C}$, there is a notion of simplicial object in $\mathbf{C}$, which is a functor $(\mathbf{\Delta})^{\mathrm{op}} \to \mathbf{C}$. Here $\mathbf{\Delta}$ is the ordinary category whose objects are sets $[n] := \{0, 1 \ldots, n\}$ $(n \geqslant 0)$, and morphisms are order-preserving morphisms of sets.

2.1.15. There is a notion of (closed) symmetric monoidal structure on a category $\mathbf{C}$. Any category admitting finite products admits a canonical symmetric monoidal structure; when equipped with this structure, we call such a category a *cartesian monoidal category*.

In a symmetric monoidal category, there is a notion of commutative monoid. We write $\mathcal{CM}on(\mathbf{C})$ for the category of commutative monoids.

Given a commutative monoid $\mathbf{O}$ in $\mathbf{C}$, there is a notion of $\mathbf{O}$-module object in $\mathbf{C}$. We write $\mathbf{O}$-*mod* for the category $\mathbf{O}$-modules.

2.1.16. A category is *contractible* if its underlying space is (weakly) contractible.

2.2. Monomorphisms.

2.2.1. Let X be a space. We say that X is *n-truncated* $(n \geqslant 0)$ if $\pi_i(\mathrm{X}, x) = 0$ for $i > n$ and any base point x. By convention, X is (-1)-*truncated* if it is empty or contractible, and (-2)-*truncated* if if is contractible.

A morphism of spaces $f : \mathrm{X} \to \mathrm{Y}$ is *n-truncated* if its fibres are n-truncated.

2.2.2. In a category $\mathbf{C}$, we say that an object c is *n-truncated* if for every object d in $\mathbf{C}$, the mapping space $\mathrm{Maps}_{\mathbf{C}}(d, c)$ is n-truncated.

A morphism $f : x \to y$ is *n-truncated* if for every object c in $\mathbf{C}$, the morphism of spaces $\mathrm{Maps}_{\mathbf{C}}(c, x) \to \mathrm{Maps}_{\mathbf{C}}(c, y)$ is n-truncated.

2.2.3. We say that a morphism f in a category $\mathbf{C}$ is a *monomorphism* if it is (-1)-truncated.

Note that a morphism is (-2)-truncated if and only if it is an isomorphism.

2.2.4. If $\mathbf{C}$ admits finite limits, then a morphism $f : x \to y$ is n-truncated ($n \geqslant -1$) if and only if the diagonal morphism $x \to x \times_y x$ is $(n-1)$-truncated.

2.2.5. The full subcategory of n-truncated objects ($n \geqslant -2$) in a category $\mathbf{C}$ is stable under small limits.

2.3. Presheaves.

2.3.1. Let $\mathbf{C}$ be a small category. A *presheaf* (of spaces) on $\mathbf{C}$ is a functor $(\mathbf{C})^{\mathrm{op}} \to \mathcal{S}pc$.

We let $\mathcal{P}(\mathbf{C})$ denote the category of presheaves on $\mathbf{C}$, which is by definition the functor category $\mathcal{F}unct((\mathbf{C})^{\mathrm{op}}, \mathcal{S}pc)$.

2.3.2. Any object $c \in \mathbf{C}$ represents a presheaf $h_{\mathbf{C}}(c)$ defined by

$$h_{\mathbf{C}}(c)(d) := \mathrm{Maps}_{\mathbf{C}}(d, c)$$

for each object $d \in \mathbf{C}$.

The assignment $c \mapsto h_{\mathbf{C}}(c)$ defines a canonical functor $h_{\mathbf{C}} : \mathbf{C} \to \mathcal{P}(\mathbf{C})$, called the Yoneda embedding. The Yoneda lemma states:

Proposition 2.3.3. *Let* F *be a presheaf on* $\mathbf{C}$*. For every object* $c \in \mathbf{C}$*, there is a canonical isomorphism of spaces*

$$\mathrm{Maps}_{\mathcal{P}(\mathbf{C})}(h_{\mathbf{C}}(c), \mathrm{F}) \to \mathrm{F}(c).$$

In particular, the Yoneda embedding $h_{\mathbf{C}}$ is fully faithful. We say that a presheaf F is *representable* if it is isomorphic to $h_{\mathbf{C}}(c)$ for some object $c \in \mathbf{C}$.

2.3.4. The category $\mathcal{P}(\mathbf{C})$ admits arbitrary small colimits and limits, and they can be computed objectwise. That is, for any diagram of presheaves $i \mapsto \mathrm{F}_i$, indexed on a category $\mathbf{I}$, we have canonical isomorphisms of spaces

$$\begin{aligned}
(\varinjlim_{i \in \mathbf{I}} \mathrm{F}_i)(c) &= \varinjlim_{i \in \mathbf{I}} \mathrm{F}_i(c),\\
(\varprojlim_{i \in \mathbf{I}} \mathrm{F}_i)(c) &= \varprojlim_{i \in \mathbf{I}} \mathrm{F}_i(c),
\end{aligned}$$

for each object $c \in \mathbf{C}$.

2.3.5. In particular, the category $\mathcal{P}(\mathbf{C})$ admits a cartesian monoidal structure, i.e. a symmetric monoidal structure where the monoidal product is given by the cartesian product.

2.3.6. The category of presheaves admits the following universal property.

Given two categories $\mathbf{C}$ and $\mathbf{D}$ where small colimits are representable, we write $\mathcal{F}unct_!(\mathbf{C}, \mathbf{D})$ for the full subcategory of $\mathcal{F}unct(\mathbf{C}, \mathbf{D})$ spanned by functors that commute with small colimits. Then we have:

Proposition 2.3.7. *Let* $\mathbf{C}$ *be a small category. For any category* $\mathbf{D}$ *where small colimits are representable, the canonical functor*

$$\mathcal{F}unct_!(\mathcal{P}(\mathbf{C}), \mathbf{D}) \to \mathcal{F}unct(\mathbf{C}, \mathbf{D}), \tag{2.1}$$

given by restriction along the Yoneda embedding, is an equivalence.

In other words, the category $\mathcal{P}(\mathbf{C})$ is freely generated by $\mathbf{C}$ under small colimits.

In particular, any functor $u : \mathbf{C} \to \mathbf{D}$ admits a canonical extension to a functor $u_! : \mathcal{P}(\mathbf{C}) \to \mathbf{D}$. This is left adjoint to the functor u^* of restriction of presheaves along u.

We call $u_!$ the *left Kan extension* of u. For any presheaf F, we have a canonical isomorphism

$$u_!(\mathrm{F}) = \varinjlim_{h_{\mathbf{C}}(c) \to \mathrm{F}} u(c), \tag{2.2}$$

where the colimit is taken over the slice category $(h_{\mathbf{C}}/\mathrm{F})$.

2.3.8. Taking $\mathbf{D} = \mathcal{P}(\mathbf{C})$ above, we see that the identity functor of $\mathcal{P}(\mathbf{C})$ is the left Kan extension of the Yoneda embedding.

In particular, every presheaf F can be canonically identified with a small colimit of representable presheaves:

$$\mathrm{F} = \varinjlim_{h_{\mathbf{C}}(c) \to \mathrm{F}} h(c). \tag{2.3}$$

2.3.9. Let $u : \mathbf{C} \to \mathbf{D}$ be a functor of small categories. Then the universal property of the category of presheaves gives a canonical functor

$$u_! : \mathcal{P}(\mathbf{C}) \to \mathcal{P}(\mathbf{D}),$$

left adjoint to the restriction functor $u^* : \mathrm{F} \mapsto \mathrm{F} \circ u$.

The functor $u_!$ is the unique functor which fits in the commutative square

$$\begin{tikzcd}
\mathbf{C} \arrow[r, "u"] \arrow[d, "h_{\mathbf{C}}"] & \mathbf{D} \arrow[d, "h_{\mathbf{D}}"] \\
\mathcal{P}(\mathbf{C}) \arrow[r, "u_!"] & \mathcal{P}(\mathbf{D})
\end{tikzcd}$$

and commutes with small colimits.

2.4. Filtered colimits.

2.4.1. Let $\mathbf{I}$ be a small category.

Definition 2.4.2. *The category* $\mathbf{I}$ *is* κ-filtered*, for a regular cardinal* κ*, if* $\mathbf{I}$*-indexed colimits commute with* κ*-small limits, in the category of spaces.*

The category $\mathbf{I}$ *is* filtered *if it is* $\aleph_0$*-filtered, i.e.* $\mathbf{I}$*-indexed colimits commute with finite limits.*

Any ordinary category is filtered in the sense of [SGA 4] if and only if it is filtered when viewed as an $(\infty, 1)$-category.

Every filtered category is *contractible*.

2.4.3. Let $\mathbf{C}$ be a small category. We define:

Definition 2.4.4. *A presheaf* F *on* $\mathbf{C}$ *is* κ-inductive *if the slice category* $(h_{\mathbf{C}}/\mathrm{F})$ *is* κ*-filtered.*

The formula (2.3) shows that a presheaf is κ-inductive if and only if it is a κ-filtered colimit of a diagram of representable presheaves.

Note that representable presheaves are κ-inductive.

In the case $\kappa = \aleph_0$, we say simply *inductive* instead of $\aleph_0$*-inductive.*

2.4.5. Let $\mathcal{I}nd_\kappa(\mathbf{C})$ (resp. $\mathcal{I}nd(\mathbf{C})$) for the full subcategory of $\mathcal{P}(\mathbf{C})$ spanned by κ-inductive presheaves (resp. by inductive presheaves).

Given two categories $\mathbf{D}$ and $\mathbf{D}'$ admitting small κ-filtered colimits, we write $\mathcal{F}unct_{\kappa\text{-filt}}(\mathbf{D}, \mathbf{D}') \subset \mathcal{F}unct(\mathbf{D}, \mathbf{D}')$ for the full subcategory spanned by functors commuting with κ-filtered colimits.

We have the following universal property for $\mathcal{I}nd_\kappa(\mathbf{C})$:

Proposition 2.4.6. *Let $\mathbf{C}$ be a small category and $\mathbf{D}$ a category admitting small κ-filtered colimits. The canonical functor*

$$\mathcal{F}unct_{\kappa\text{-filt}}(\mathcal{I}nd_\kappa(\mathbf{C}), \mathbf{D}) \to \mathcal{F}unct(\mathbf{C}, \mathbf{D}),$$

given by restriction along the Yoneda embedding, is an equivalence.

In particular any functor $u : \mathbf{C} \to \mathbf{D}$ extends uniquely to a functor $u_! : \mathcal{I}nd_\kappa(\mathbf{C}) \to \mathbf{D}$. This is also given by the formula (2.2), and we also call it the left Kan extension of u.

2.5. Sifted colimits.

2.5.1. Let $\mathbf{I}$ be a small category.

Definition 2.5.2. *(i) The category $\mathbf{I}$ is* κ-sifted*, for a regular cardinal κ, if $\mathbf{I}$-indexed colimits commute with κ-products, in the category of spaces.*

(ii) The category $\mathbf{I}$ is sifted *if it is $\aleph_0$-sifted, i.e. $\mathbf{I}$-indexed colimits commute with finite products.*

Any κ-filtered category is κ-sifted. The category $\mathbf{\Delta}^{\mathrm{op}}$ is sifted.

Any ordinary category is sifted in the sense of [GU71] if and only if it is sifted when viewed as an $(\infty, 1)$-category.

Every sifted category is *contractible*.

2.5.3. Let $\mathbf{C}$ be a small category. We define:

Definition 2.5.4. *A presheaf F on $\mathbf{C}$ is* weakly κ-inductive *if the slice category $(h_{\mathbf{C}}/\mathrm{F})$ is κ-sifted.*

The formula (2.3) shows that a presheaf is weakly κ-inductive if and only if it is a κ-sifted colimit of a diagram of representable presheaves. If $\mathbf{C}$ admits κ-small coproducts, a presheaf is weakly κ-inductive if and only if it commutes with κ-small products.

Note that representable presheaves are weakly κ-inductive.

In the case $\kappa = \aleph_0$, we say simply *weakly inductive* instead of *weakly $\aleph_0$-inductive*.

2.5.5. Let $\mathcal{W}ind_\kappa(\mathbf{C})$ (resp. $\mathcal{W}ind(\mathbf{C})$) for the full subcategory of $\mathcal{P}(\mathbf{C})$ spanned by weakly κ-inductive presheaves (resp. by weakly inductive presheaves).

Given two categories $\mathbf{D}$ and $\mathbf{D}'$ admitting small κ-sifted colimits, we write $\mathcal{F}unct_{\kappa\text{-sift}}(\mathbf{D}, \mathbf{D}') \subset \mathcal{F}unct(\mathbf{D}, \mathbf{D}')$ for the full subcategory spanned by functors commuting with κ-sifted colimits.

We have the following universal property for $\mathcal{W}ind_\kappa(\mathbf{C})$:

Proposition 2.5.6. *Let $\mathbf{C}$ be a small category and $\mathbf{D}$ a category admitting small κ-sifted colimits. The canonical functor*

$$\mathcal{F}unct_{\kappa\text{-sift}}(\mathcal{W}ind_\kappa(\mathbf{C}), \mathbf{D}) \to \mathcal{F}unct(\mathbf{C}, \mathbf{D}),$$

given by restriction along the Yoneda embedding, is an equivalence.

In particular any functor $u : \mathbf{C} \to \mathbf{D}$ extends uniquely to a functor $u_! : \mathcal{W}ind_\kappa(\mathbf{C}) \to \mathbf{D}$. This is also given by the formula (2.2), and we also call it the left Kan extension of u.

2.6. Arenas. The terminology "arena" was introduced by [Joy04]. In [Lur09b] the term "presentable ∞-category" is used instead.

2.6.1. Let $\mathbf{C}$ and $\mathbf{D}$ be categories where small colimits are representable, and κ a regular cardinal. A functor $\mathbf{C} \to \mathbf{D}$ is *κ-accessible* if it commutes with κ-filtered colimits. It is *accessible* if it is κ-accessible for some κ.

Definition 2.6.2. *A* left localization *of a category $\mathbf{C}$ is a functor $\gamma : \mathbf{C} \to \mathbf{D}$ admitting a fully faithful right adjoint.*

An accessible localization *(resp. κ-accessible localization) is a left localization such that the right adjoint of γ is accessible (resp. κ-accessible).*

An exact localization *is an accessible localization such that γ is exact (i.e. commutes with finite limits).*

2.6.3. Let $\mathbf{C}$ be a category. We define:

Definition 2.6.4. *The category $\mathbf{C}$ is* compactly generated *if there exists a small category $\mathbf{C}_0$ and an $\aleph_0$-accessible localization $\gamma : \mathcal{P}(\mathbf{C}_0) \to \mathbf{C}$.*

For example, for any small category $\mathbf{C}_0$, the category $\mathcal{I}nd(\mathbf{C}_0)$ is compactly generated.

2.6.5. Let $\mathbf{C}$ be a category. The following notion will play an important role throughout this text:

Definition 2.6.6. *The category $\mathbf{C}$ is an* arena[6] *if there exists a small category $\mathbf{C}_0$ and an accessible localization $\gamma : \mathcal{P}(\mathbf{C}_0) \to \mathbf{C}$.*

A morphism of arenas *is a functor that commutes with small colimits.*

We write $\mathcal{A}rena$ for the category of arenas.

2.6.7. The class of arenas is stable under the formation of slice categories and functor categories:

Lemma 2.6.8. *Let $\mathbf{C}$ be an arena. For any functor $p : \mathbf{I} \to \mathbf{C}$, the slice categories $(\mathbf{C}/p)$ and $(p\backslash\mathbf{C})$ are arenas.*

Lemma 2.6.9. *Let $\mathbf{C}$ and $\mathbf{D}$ be arenas. Then the category $\mathcal{F}unct_!(\mathbf{C}, \mathbf{D})$ of morphisms of arenas is an arena.*

2.6.10. The following fact will be referred to as the "adjoint functor theorem":

Proposition 2.6.11. *Let $\mathbf{C}$ and $\mathbf{D}$ be arenas. Then a functor $u : \mathbf{C} \to \mathbf{D}$ admits a right adjoint if and only if it commutes with small colimits. It admits a left adjoint if and only if commutes with small limits and is accessible.*

We also have:

Proposition 2.6.12. *Let $\mathbf{C}$ be an arena. Then a functor $(\mathbf{C})^{\mathrm{op}} \to \mathcal{S}pc$ is representable if and only if it commutes with small limits.*

[6] This term is due to Joyal [Joy04]. The term *presentable* or *locally presentable* $(\infty,1)$*-category* is more common in the literature.

2.6.13. Let S be a set of morphisms in an arena $\mathbf{C}$. We define:

Definition 2.6.14. *(i) An object* $c \in \mathbf{C}$ *is* S-local *if, for every morphism* $f : x \to y$ *in* S*, the induced morphism of spaces*

$$\mathrm{Maps}_{\mathbf{C}}(y, c) \to \mathrm{Maps}_{\mathbf{C}}(x, c)$$

is invertible.

(ii) A morphism $f : x \to y$ *is an* S-local equivalence *if for every* S*-local object* c*, the induced morphism of spaces*

$$\mathrm{Maps}_{\mathbf{C}}(y, c) \to \mathrm{Maps}_{\mathbf{C}}(x, c)$$

is invertible.

We have:

Proposition 2.6.15. *Let* $\mathbf{C}$ *be an arena. For any essentially small*[7] *set* S *of morphisms in* $\mathbf{C}$*, the inclusion of the full subcategory* $\mathbf{C}_{\mathrm{S}}$ *of* S*-local objects admits a left adjoint* $\mathrm{L} : \mathbf{C} \to \mathbf{C}_{\mathrm{S}}$*, which exhibits* $\mathbf{C}_{\mathrm{S}}$ *as an accessible left localization of* $\mathbf{C}$*.*

In the above situation, a morphism f in $\mathbf{C}$ induces an isomorphism $\mathrm{L}(f)$ if and only if f is an S-local equivalence.

In fact, all accessible left localizations of an arena $\mathbf{C}$ arise in the above way.

2.6.16. Let $\mathbf{C}$ be an arena and S an essentially small set of morphisms.

Given an arena $\mathbf{D}$, let $\mathcal{F}unct_{!,\mathrm{S}}(\mathbf{C}, \mathbf{D})$ denote the full subcategory of $\mathcal{F}unct_{!}(\mathbf{C}, \mathbf{D})$ spanned by functors that send morphisms in S to isomorphisms in $\mathbf{D}$.

We have the following universal property of $\mathbf{C}_{\mathrm{S}}$:

Proposition 2.6.17. *For any arena* $\mathbf{D}$*, the canonical morphism*

$$\mathcal{F}unct_{!}(\mathbf{C}_{\mathrm{S}}, \mathbf{D}) \xrightarrow{\sim} \mathcal{F}unct_{!,\mathrm{S}}(\mathbf{C}, \mathbf{D})$$

given by restriction along the functor $\mathrm{L} : \mathbf{C} \to \mathbf{C}_{\mathrm{S}}$*, is an equivalence.*

2.6.18. Another way to formulate the above is as follows.

Consider the category of pairs $(\mathbf{C}, \mathrm{S})$, with $\mathbf{C}$ an arena and S an essentially small set of morphisms. There is a canonical fully faithful functor $\mathbf{D} \mapsto (\mathbf{D}, \mathrm{iso})$, where iso is the set of isomorphisms in $\mathbf{D}$.

It admits a right adjoint, the forgetful functor $(\mathbf{C}, \mathrm{S}) \mapsto \mathbf{C}$.

It also admits a left adjoint, the left localization functor $(\mathbf{C}, \mathrm{S}) \mapsto \mathbf{C}_{\mathrm{S}}$.

Further, both adjoints are symmetric monoidal, with respect to the cartesian monoidal structure on the category of pairs; see [Rob14, §9.1].

2.7. Module arenas.

[7] A set of morphisms S is *essentially small* if there is a small subset $\mathrm{S}_0 \subset \mathrm{S}$ such that every morphism in S is isomorphic to a morphism of S_0 in the category $\mathcal{A}rrows(\mathbf{C})$.

2.7.1. The category of arenas admits a canonical symmetric monoidal structure.

Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be arenas. There is a canonical functor

$$\mathbf{C}_1 \times \mathbf{C}_2 \to \mathbf{C}_1 \otimes \mathbf{C}_2$$

which commutes with small colimits in each argument, and for any arena $\mathbf{D}$, the canonical functor

$$\mathcal{F}unct_!(\mathbf{C}_1 \otimes \mathbf{C}_2, \mathbf{D}) \to \mathcal{F}unct(\mathbf{C}_1 \times \mathbf{C}_2, \mathbf{D})$$

is fully faithful with essential image spanned by functors $\mathbf{C}_1 \times \mathbf{C}_2 \to \mathbf{D}$ that commute with small colimits in each argument.

2.7.2. A *symmetric monoidal arena* is a commutative monoid object in the category of arenas.

Equivalently, this is a symmetric monoidal category whose underlying category is an arena, and such that the monoidal product commutes with small colimits in each variable.

For example, given a small category $\mathbf{C}$, the category $\mathcal{P}(\mathbf{C})$ is a cartesian monoidal arena, i.e. a symmetric monoidal arena whose monoidal product is given by the cartesian product.

2.7.3. We will write $\mathcal{A}renamon$ for the category of symmetric monoidal arenas.

2.7.4. By the adjoint functor theorem, the symmetric monoidal structure on an arena is automatically *closed*, i.e. there exists a bifunctor $\underline{\mathrm{Hom}}_{\mathbf{C}}(-,-)$ which is right adjoint to $- \otimes -$.

2.7.5. Let $\mathbf{O}$ be a symmetric monoidal arena. An $\mathbf{O}$-*module arena* is an $\mathbf{O}$-module in the category of arenas. We let $\mathbf{O}$-*mod* denote the category of $\mathbf{O}$-module arenas.

A morphism of $\mathbf{O}$-modules $u : \mathbf{M} \to \mathbf{N}$ in particular gives rise to functorial isomorphisms

$$o \otimes u(n) = u(o \otimes m)$$

for any objects $o \in \mathbf{O}$, $m \in \mathbf{M}$, $n \in \mathbf{N}$ (but also contains much more coherence data).

2.7.6. One can define a notion of *lax* (resp. *colax*) functor of $\mathbf{O}$-modules. This essentially amounts to requiring that, instead of isomorphisms as above, we have morphisms $o \otimes u(n) \to u(o \otimes m)$ (resp. $u(o \otimes m) \to o \otimes u(n)$).

The following lemma will be useful:

Lemma 2.7.7. *Let $\mathbf{O}$ be a symmetric monoidal arena. Let $\mathbf{M}$ and $\mathbf{N}$ be $\mathbf{O}$-module arenas. Suppose that there is an adjunction*

$$u : \mathbf{M} \to \mathbf{N}, \qquad v : \mathbf{N} \to \mathbf{M}$$

of underlying categories. Then the structure of colax functor of $\mathbf{O}$-modules on u is equivalent to the structure of lax functor of $\mathbf{O}$-modules on v.

2.8. Toposes.

2.8.1. An important subclass of arenas is formed by *toposes*:

Definition 2.8.2. *A* topos *is a category $\mathbf{X}$ such that there exists a small category $\mathbf{C}$ and an* exact *localization $\gamma : \mathcal{P}(\mathbf{C}) \to \mathbf{X}$.*

A morphism of toposes *is a functor admitting an exact left adjoint.*

Any topos has the property of *universality of colimits*:

Proposition 2.8.3. *For any morphism $f : x \to y$, the functor $\mathbf{X}_{/y} \to \mathbf{X}_{/x}$ given by the assignment $(y' \to y) \mapsto (y' \times_y x \to x)$ commutes with small colimits.*

2.8.4. Let $f : x \to y$ be a morphism in a topos $\mathbf{X}$.

The *Čech nerve* of f is the simplicial object of $\mathbf{X}_{/y}$ defined degreewise as the $(n+1)$-fold fibred product

$$\check{\mathrm{C}}(x/y)_n = x \underset{y}{\times} x \underset{y}{\times} \cdots \underset{y}{\times} x.$$

More precisely, the functor $f \mapsto \check{\mathrm{C}}(x/y)_\bullet$ on the slice category $\mathbf{X}_{/y}$ is right adjoint to the functor that evaluates a simplicial object at degree zero.

We define:

Definition 2.8.5. *The morphism $f : x \to y$ is an* effective epimorphism *if the canonical morphism*

$$\varinjlim_{n \in \mathbf{\Delta}^{\mathrm{op}}} \check{\mathrm{C}}(x/y)_n \to e$$

is invertible, where e denotes the terminal object of $\mathbf{X}$.

2.8.6. Let $\mathbf{X}$ be a topos. Every morphism $f : x \to y$ is (-2)-*connected*. It is (-1)-*connected* if it is an effective epimorphism. It is n-*connected* if it is an effective epimorphism and its diagonal morphism $x \to x \times_y x$ is $(n-1)$-connected.

2.8.7. In a topos $\mathbf{X}$, the classes of n-connected and n-truncated morphisms form an orthogonal factorization system, for each $n \geqslant -2$. In particular, every morphism f in $\mathbf{X}$ admits a factorization of the form $f = p \circ i$, where i is n-truncated and p is n-connected.

Further, we have:

Lemma 2.8.8. *Let $\mathbf{X}$ be a topos. A morphism $f : x \to y$ in $\mathbf{X}$ is invertible if and only if it is both n-truncated and n-connected, for some $n \geqslant -2$.*

2.9. Sites.

2.9.1. Recall the notion of Grothendieck (pre)topologies on ordinary categories from [SGA 4]. These can be adapted to the setting of $(\infty, 1)$-categories. In fact, a topology on a category $\mathbf{C}$ is equivalent to a topology on its underlying ordinary category $(\mathbf{C})^{\mathrm{ordn}}$.

A *site* is a category equipped with a Grothendieck topology.

2.9.2. Given a topology τ on a small category $\mathbf{C}$, let S_τ denote the (small) set of morphisms

$$\mathrm{R} \hookrightarrow h_{\mathbf{C}}(c)$$

for c an object of $\mathbf{C}$ and $\mathrm{R} \hookrightarrow h_{\mathbf{C}}(c)$ a τ-covering sieve.

A τ-*sheaf* on $\mathbf{C}$ is by definition a presheaf F which is S_τ-local. That is, for each τ-covering sieve $\mathrm{R} \hookrightarrow h_{\mathbf{C}}(c)$, the canonical morphism of spaces

$$\mathrm{F}(c) = \mathrm{Maps}(h_{\mathbf{C}}(c), \mathrm{F}) \to \mathrm{Maps}(\mathrm{R}, \mathrm{F}) \tag{2.4}$$

is invertible. (We also refer to this property as τ-*descent*.)

We say that a morphism of presheaves is a τ-*local equivalence* if it is an S_τ-local equivalence.

2.9.3. We write $\mathcal{S}h_\tau(\mathbf{C})$ for the full subcategory of τ-sheaves. By [Chap. 0, Proposition 2.6.15] we obtain that $\mathcal{S}h_\tau(\mathbf{C})$ is an accessible localization of the arena $\mathcal{P}(\mathbf{C})$. In fact, this is an *exact* localization (hence a topos).

2.9.4. If the topology τ is generated by a *pretopology*, then the condition of τ-descent can be phrased equivalently as follows: for every τ-covering family $(f_\alpha : c_\alpha \to c)_\alpha$, the canonical morphism of spaces

$$\mathrm{F}(c) \to \varprojlim_{n\in\mathbf{\Delta}} \Gamma(\check{\mathrm{C}}(c_\alpha/c)_n, \mathrm{F}) \tag{2.5}$$

is invertible.

Here $\check{\mathrm{C}}(c_\alpha/c)_\bullet$ denotes the Čech nerve of the morphism $\sqcup_\alpha c_\alpha \to c$.

2.10. Stable categories.

2.10.1. Let $\mathbf{C}$ be a category. We say that $\mathbf{C}$ is *pointed* if it admits a *zero object*, i.e. an object which is both terminal and initial.

2.10.2. Let $\mathbf{C}$ be a pointed category admitting finite colimits and limits. For any object c in $\mathbf{C}$, we can form its *suspension* object

$$\Sigma(x) := 0 \underset{x}{\sqcup} 0,$$

and its *loop space* object

$$\Omega(x) := 0 \underset{x}{\times} 0,$$

where 0 is a zero object.

The functors (Σ, Ω) form an adjoint pair; if it is an equivalence, then we say that the category $\mathbf{C}$ is *stable*.

In this case, we will write $\mathcal{F}[1] := \Sigma(\mathcal{F})$ and $\mathcal{F}[-1] := \Omega(\mathcal{F})$.

2.10.3. Equivalently, a category $\mathbf{C}$ as above is stable if any commutative square in $\mathbf{C}$ is cartesian if and only if it is cocartesian.

2.10.4. The most fundamental example of a stable category is $\mathcal{S}pt$, the category of spectra. This is obtained from the category of spaces by a formal procedure called stabilization. In particular, there is a canonical adjoint pair

$$\Sigma^\infty : \mathcal{S}pc \to \mathcal{S}pt, \qquad \Omega^\infty : \mathcal{S}pt \to \mathcal{S}pc.$$

2.10.5. There is a canonical t-structure on the category of spectra.

A spectrum E is *n-connected* (resp. *n-connective*) if $\pi_i(\mathrm{E}) = 0$ for each $i \leqslant n$ (resp. for each $i < n$). It is *n-coconnected* (resp. *n-coconnective*) if $\pi_i(\mathrm{E}) = 0$ for each $i \geqslant n$ (resp. for each $i > n$).

We will abbreviate the term 0-connective (= (-1)-connected) to *connective*, and 0-coconnective (= 1-coconnected) to *coconnective*.

2.10.6. Commutative monoids in the category $\mathcal{S}pt$ are $\mathbf{E}_\infty$-ring spectra.

(We will not use this notion in this text.)

3. $(\infty, 2)$-CATEGORIES

The references for this section are [GR16] and [Lur09c].

3.1. $(\infty, 2)$-categories as complete Segal spaces. We will use complete Segal spaces in $(\infty, 1)$-$\mathcal{C}at$ to model $(\infty, 2)$-categories.

3.1.1. An $(\infty, 2)$-*category* $\mathbb{C}$ is a complete Segal space in the $(\infty, 1)$-category of $(\infty, 1)$-categories. That is, it is the datum of a simplicial object $\mathrm{Seq}_\bullet(\mathbb{C})$ in the $(\infty, 1)$-category of $(\infty, 1)$-categories satisfying the following conditions:

(1) The $(\infty, 1)$-category $\mathrm{Seq}_0(\mathbb{C})$ is an ∞-groupoid.

(2) The canonical functor $\mathrm{Seq}_{m+n}(\mathbb{C}) \to \mathrm{Seq}_m(\mathbb{C}) \times_{\mathrm{Seq}_0(\mathbb{C})} \mathrm{Seq}_n(\mathbb{C})$ is an equivalence.

(3) The Segal space $(\mathrm{Seq}_\bullet(\mathbb{C}))^{\mathcal{S}pc}$ is *complete* (see [GR16, Chap. A.1]).

We will write $(\infty, 2)$-$\mathcal{C}at$ for the $(\infty, 1)$-category of $(\infty, 2)$-categories.

3.1.2. There are other models for the $(\infty, 1)$-category of $(\infty, 2)$-categories.

For example, the $(\infty, 1)$-category $(\infty, 2)$-$\mathcal{C}at$ is equivalent to the $(\infty, 1)$-category of 2-fold complete Segal spaces. This equivalence is induced by an equivalence between the $(\infty, 1)$-category of $(\infty, 1)$-categories and that of 1-fold complete Segal spaces (i.e. complete Segal spaces in the $(\infty, 1)$-category of spaces); see [JT07].

It follows, according to [Hau15], that it is also equivalent to the $(\infty, 1)$-category of $(\infty, 1)$-categories enriched in $(\infty, 1)$-$\mathcal{C}at$.

3.1.3. Any ordinary 2-category can be viewed as an $(\infty, 2)$-category where $\mathrm{Seq}_1(\mathbb{C})$ is an ordinary category. This defines a fully faithful embedding from the $(\infty, 1)$-category of ordinary 2-categories to the $(\infty, 1)$-category of $(\infty, 2)$-categories.

Conversely, any $(\infty, 2)$-category gives rise to an ordinary 2-category $(\mathbf{C})^{\text{2-ordn}}$. This defines a left adjoint to the above embedding.

3.1.4. Given an $(\infty, 2)$-category $\mathbb{C}$, there is an $(\infty, 2)$-category $(\mathbb{C})^{\text{1-op}}$ obtained by reversing the directions of 1-morphisms, an $(\infty, 2)$-category $(\mathbb{C})^{\text{2-op}}$ obtained by reversing the directions of 2-morphisms, and an $(\infty, 2)$-category $(\mathbb{C})^{\text{1\&2-op}}$ obtained by reversing the directions of both 1- and 2-morphisms.

3.1.5. There is an $(\infty, 2)$-category $(\infty, \mathbf{1})$-$\mathbf{Cat}$ of $(\infty, 1)$-categories, whose underlying $(\infty, 1)$-category coincides with $(\infty, 1)$-$\mathcal{C}at$.

3.1.6. In the sequel, the term "2-category" will mean $(\infty, 2)$-category by default; when we need to refer to the classical notion of 2-category, we will say "$(2, 2)$-category".

3.2. Passage to right/left adjoints.

3.2.1. Recall that in an 2-category $\mathbb{C}$, there is a notion of adjunction between two objects x and y.

A pair $(f : x \to y, g : y \to x)$ forms an adjunction if and only if it defines an adjunction in the underlying ordinary 2-category $(\mathbb{C})^{\text{2-ordn}}$.

3.2.2. Let $\mathbb{C}$ and $\mathbb{D}$ be 2-categories. We say that a 2-functor $u : \mathbb{C} \to \mathbb{D}$ is *right-adjointable* (resp. *left-adjointable*) if, for each 1-morphism $f : x \to y$ in $\mathbb{C}$, its image $u(f)$ admits a right adjoint (resp. a left adjoint) in the 2-category $\mathbb{D}$.

3.2.3. Let $\mathrm{Maps}_!(\mathbb{S}, \mathbb{T})$ denote the space of right-adjointable 2-functors. Let $\mathrm{Maps}_*(\mathbb{S}, \mathbb{T})$ denote the space of left-adjointable 2-functors.

Lemma 3.2.4. *There is a canonical isomorphism of spaces*

$$\mathrm{Maps}_!(\mathbb{S}, \mathbb{T}) = \mathrm{Maps}_*((\mathbb{S})^{1\&2\text{-op}}, \mathbb{T}).$$

Given a right-adjointable 2-functor $u : \mathbb{S} \to \mathbb{T}$, we will call the corresponding functor $u_* : (\mathbb{S})^{1\&2\text{-op}} \to \mathbb{T}$ the functor obtained from u by *passage to right adjoints.*

Dually, given a left-adjointable 2-functor $u : \mathbb{S} \to \mathbb{T}$, we will call the corresponding functor $u_! : (\mathbb{S})^{1\&2\text{-op}} \to \mathbb{T}$ the functor obtained from u by *passage to left adjoints.*

3.3. Adjointable squares. In this section we will formulate the notion of *horizontally/vertically left/right-adjointable square* in a 2-category, which we will be used in the text to express base change formulas.

3.3.1. We fix an $(\infty, 2)$-category $\mathbb{C}$.

Let Θ be a square in $\mathbb{C}$

$$\begin{tikzcd} \mathbf{C} \arrow[r, "u"] \arrow[d, "v"] & \mathbf{C}' \arrow[d, "v'"] \\ \mathbf{D} \arrow[r, "u'"] & \mathbf{D}' \end{tikzcd} \tag{3.1}$$

which commutes up to an invertible 2-morphism

$$v'u \xrightarrow{\sim} u'v.$$

Suppose that v (resp. v') admits a right adjoint v^{R} (resp. $(v')^{\mathrm{R}}$) in $\mathbb{C}$. Then the square $\Theta^{vert:\mathrm{R}}$

$$\begin{tikzcd} \mathbf{C} \arrow[r, "u"] & \mathbf{C}' \\ \mathbf{D} \arrow[r, "u'"] \arrow[u, "v^{\mathrm{R}}"] & \mathbf{D}' \arrow[u, "(v')^{\mathrm{R}}"'] \end{tikzcd} \tag{3.2}$$

commutes up to the 2-morphism

$$uv^{\mathrm{R}} \to (v')^{\mathrm{R}}v'uv^{\mathrm{R}} \xrightarrow{\sim} (v')^{\mathrm{R}}u'vv^{\mathrm{R}} \to (v')^{\mathrm{R}}u', \tag{3.3}$$

where the first morphism is obtained by precomposition with the counit of the adjunction, the isomorphism in the middle is given by the commutativity of the square Θ, and the final morphism is given by the unit of the adjunction.

If this 2-morphism is invertible, then we say that the square Θ is *vertically right-adjointable.*

3.3.2. Similarly if v (resp. v') admits a *left* adjoint v^{L} (resp. $(v')^{\mathrm{L}}$), then the square $\Theta^{vert:\mathrm{L}}$

$$\begin{tikzcd} \mathbf{C} \arrow[r, "u"] & \mathbf{C}' \\ \mathbf{D} \arrow[r, "u'"] \arrow[u, "v^{\mathrm{L}}"] & \mathbf{D}' \arrow[u, "(v')^{\mathrm{L}}"'] \end{tikzcd} \tag{3.4}$$

commutes up to the 2-morphism

$$(v')^{\mathrm{L}}u' \to (v')^{\mathrm{L}}u'vv^{\mathrm{L}} \xleftarrow{\sim} (v')^{\mathrm{L}}v'uv^{\mathrm{L}} \to uv^{\mathrm{L}}. \tag{3.5}$$

If this is invertible, we say that the square Θ is *vertically left-adjointable.*

3.3.3. If u (resp. u') admits a right adjoint u^{R} (resp. $(u')^{\mathrm{R}}$), then the square $\Theta^{horiz:\mathrm{R}}$

$$\begin{tikzcd} \mathbf{C} & \mathbf{C}' \arrow[l, "u^{\mathrm{R}}"'] \arrow[d, "v'"] \\ \mathbf{D} \arrow[from=u, "v"'] & \mathbf{D} \arrow[l, "(u')^{\mathrm{R}}"] \end{tikzcd} \tag{3.6}$$

commutes up to a 2-morphism

$$vu^{\mathrm{R}} \to (u')^{\mathrm{R}}v'. \tag{3.7}$$

If it is invertible, we say that Θ is *horizontally right-adjointable.*

Similarly, if u (resp. u') admits a left adjoint u^{L} (resp. $(u')^{\mathrm{L}}$), then the square $\Theta^{horiz:\mathrm{L}}$

$$\begin{tikzcd} \mathbf{C} & \mathbf{C}' \arrow[l, "u^{\mathrm{L}}"'] \arrow[d, "v'"] \\ \mathbf{D} \arrow[from=u, "v"'] & \mathbf{D} \arrow[l, "(u')^{\mathrm{L}}"] \end{tikzcd} \tag{3.8}$$

commutes up to a 2-morphism

$$(u')^{\mathrm{L}}v' \to vu^{\mathrm{L}}. \tag{3.9}$$

If it is invertible, we say that Θ is *horizontally left-adjointable.*

3.3.4. We have:

Lemma 3.3.5. *Suppose that in the square Θ (3.1), u (resp. u') admits a right adjoint u^{R} (resp. $(u')^{\mathrm{R}}$), and v (resp. v') admits a left adjoint v^{L} (resp. $(v')^{\mathrm{L}}$). Then Θ is vertically left-adjointable if and only if it is horizontally right-adjointable.*

Proof. The square $\Theta^{vert:\mathrm{L}}$ (resp. $\Theta^{horiz:\mathrm{R}}$) commutes up to a 2-morphism $\alpha : (v')^{\mathrm{L}}u' \to uv^{\mathrm{L}}$ (resp. $\beta : vu^{\mathrm{R}} \to (u')^{\mathrm{R}}v'$). The category of left adjoint functors $\mathbf{C} \to \mathbf{D}$ is equivalent to the category of right adjoint functors $\mathbf{D} \to \mathbf{C}$ (see [GR16, Chap. A.3, Cor. 3.1.9]), and under this equivalence the morphism α corresponds to the morphism β. $\square$

3.4. The 2-category of arena modules.

3.4.1. Let $\mathcal{A}renamod$ denote the closed symmetric monoidal $(\infty, 1)$-category of pairs $(\mathbf{O}, \mathbf{C})$ with $\mathbf{O}$ a symmetric monoidal arena and $\mathbf{C}$ an $\mathbf{O}$-module arena.

We refer to [Lur16, Def. 3.3.3.8], where it is defined as an ∞-operad, [Lur16, Thm. 4.5.2.1], where it is shown to be a symmetric monoidal $(\infty, 1)$-category, and [Lur16, Cor. 4.4.2.15], where its symmetric monoidal structure is shown to be closed.

3.4.2. There is a canonical functor

$$\mathcal{A}rrows(\mathcal{A}renamon) \to \mathcal{A}renamod \tag{3.10}$$

defined on objects by the assignment

$$(u : \mathbf{O} \to \mathbf{O}') \mapsto (\mathbf{O}, \mathbf{O}'),$$

where $\mathbf{O}'$ is viewed as an $\mathbf{O}$-module via the symmetric monoidal functor u.

3.4.3. Let **Arenamod** denote the $(\infty, 2)$-category described informally as follows:

Its objects are the same as those of $\mathcal{A}renamod$, i.e. pairs $(\mathbf{O}, \mathbf{C})$ with $\mathbf{O}$ a symmetric monoidal arena and $\mathbf{C}$ an $\mathbf{O}$-module arena.

Its 1-morphisms $(\mathbf{O}, \mathbf{C}) \to (\mathbf{O}', \mathbf{C}')$ are the same as those of $\mathcal{A}renamod$, i.e. pairs (u, v) with $u : \mathbf{O} \to \mathbf{O}'$ a symmetric monoidal morphism of arenas and $v : \mathbf{C} \to \mathbf{C}'$ a morphism of $\mathbf{O}$-module categories, where $\mathbf{C}'$ is viewed as a $\mathbf{O}$-module via the functor u.

Its 2-morphisms $(u, v) \to (u', v')$ are pairs (α, β) with $\alpha : u \to u'$ a symmetric monoidal natural transformation, and $\beta : v \to v'$ an $\mathbf{O}$-linear natural transformation.

We refer to [Hau14] for the precise definition as a 2-fold complete Segal space.

4. Derived schemes

In this section we give the definition of derived schemes, following the functorial approach of Toën–Vezzosi [TV08] [MT10]. It is also possible to give a definition using locally ringed $(\infty, 1)$-toposes, in line with the definition of classical schemes based on locally ringed toposes; see [Lur09a].

4.1. Simplicial commutative algebra.

4.1.1. Consider the full subcategory $\mathcal{P}oly$ of the category of (small) commutative rings, spanned by the polynomial algebras $\mathbf{Z}[\mathrm{T}_1, \ldots, \mathrm{T}_n]$ $(n \geqslant 0)$.

Definition 4.1.2. *A* simplicial commutative ring *is a weakly inductive presheaf of spaces on* $\mathcal{P}oly$.

In other words, it is a presheaf which sends finite coproducts in $\mathcal{P}oly$ to products; see [Chap. 0, Paragraph 2.5].

Let $\mathcal{SCR}ing$ denote the category of simplicial commutative rings, a full subcategory of the category of presheaves on $\mathcal{P}oly$. Recall from [Chap. 0, Proposition 2.5.6] that this is the free completion of $\mathcal{P}oly$ by sifted colimits; in particular, it is an arena.

Remark 4.1.3. One can show that $\mathcal{SCR}ing$ is the $(\infty, 1)$-category obtained from the ordinary category of simplicial objects in the category of commutative rings, by inverting weak homotopy equivalences (i.e. taking an $(\infty, 1)$-categorical localization). See [Lur09a, Rmk. 4.1.2].

4.1.4. There is a canonical functor $\mathrm{A} \mapsto \mathrm{A}_{\mathcal{S}pc}$, sending a simplicial commutative ring to its underlying space.

This is defined by evaluation on the polynomial ring $\mathbf{Z}[\mathrm{T}]$, i.e. $\mathrm{A}_{\mathcal{S}pc} := \mathrm{A}(\mathbf{Z}[\mathrm{T}])$. Note that this is conservative.

The space $\mathrm{A}_{\mathcal{S}pc}$ has a canonical base point $\mathrm{pt} \to \mathrm{A}_{\mathcal{S}pc}$, induced by the canonical homomorphism $\mathbf{Z} \to \mathbf{Z}[\mathrm{T}]$. We write $\mathrm{H}^{-n}(\mathrm{A})$ for its nth homotopy group. There is a canonical structure of (ordinary) commutative ring on $\mathrm{H}^0(\mathrm{A})$, and of $\mathrm{H}^0(\mathrm{A})$-module on each $\mathrm{H}^{-n}(\mathrm{A})$.

4.1.5. We also have the Eilenberg–Mac Lane functor $\mathrm{A} \mapsto \mathrm{A}_{\mathcal{S}pt}$, sending a simplicial commutative ring to its associated (connective) $\mathbf{E}_\infty$-ring spectrum.

This is obtained from the universal property of $\mathcal{SCR}ing$ (see [Chap. 0, Proposition 2.5.6]). Hence it is characterized as the unique functor that commutes with sifted colimits and sends a polynomial ring $\mathrm{R} = \mathbf{Z}[\mathrm{T}_1, \ldots, \mathrm{T}_n]$ to its Eilenberg–Mac Lane spectrum $\mathrm{R}_{\mathcal{S}pt}$. In fact, it commutes with arbitrary small colimits and limits, and is conservative.

The underlying space of the spectrum $\mathrm{A}_{\mathcal{S}pt}$ coincides with $\mathrm{A}_{\mathcal{S}pc}$. The spectrum $\mathrm{A}_{\mathcal{S}pt}$ is connective (i.e. $\mathrm{H}^n(\mathrm{A}_{\mathcal{S}pt}) = 0$ for $n > 0$).

4.1.6. Given a simplicial commutative ring A, an A-*module* is a (left) module over the $\mathbf{E}_\infty$-ring spectrum $\mathrm{A}_{\mathcal{S}pt}$. We write A-*mod* for the category of A-modules, which is a stable symmetric monoidal arena.

We denote $\mathrm{H}^{-i}(\mathrm{M}) := \pi_i(\mathrm{M})$ for each i.

Remark 4.1.7. If A is an ordinary commutative ring, then the underlying $(1, 1)$-category of A-*mod* coincides with the derived category of the abelian category of A-modules.

We let A-*alg* denote the category of A-algebras, i.e. the slice category ${}_{\mathrm{A}\backslash}\mathcal{SCR}ing$.

4.1.8. Note that if we took presheaves of *sets* in [Chap. 0, Definition 4.1.2], we would recover the category of (ordinary) commutative rings. In particular, the 0-truncated simplicial commutative rings are precisely ordinary commutative rings. The 0-truncation functor, left adjoint to the inclusion of ordinary commutative rings, is nothing else than the functor $A \mapsto H^0(A)$.

4.1.9. Let A be a simplicial commutative ring. For any connective A-module M, we will write $\mathrm{Sym}_A(M)$ for the free A-algebra generated by M. That is, we have a canonical isomorphism

$$\mathrm{Maps}_{A\text{-}alg}(\mathrm{Sym}_A(M), B) \xrightarrow{\sim} \mathrm{Maps}_{A\text{-}mod}(M, B)$$

for each A-algebra B.

4.2. Prestacks.

4.2.1. A *prestack* is a presheaf of spaces on the category $(\mathcal{SCR}ing)^{\mathrm{op}}$, i.e. a functor $\mathcal{SCR}ing \to \mathcal{S}pc$ (see [Chap. 0, Paragraph 2.6]).

We write $\mathcal{P}restk$ for the category of prestacks.

4.2.2. Given a simplicial commutative ring A, we will write Spec(A) for the prestack represented by A. We say that a prestack is an *affine scheme* if it is represented by a simplicial commutative ring, and write $\mathcal{S}ch_{\mathrm{aff}}$ for the full subcategory of $\mathcal{P}restk$ spanned by affine schemes.

Let S be a prestack. For a simplicial commutative ring A, we say that an A-*point of* S is a morphism $s : \mathrm{Spec}(A) \to S$, or equivalently a point of the space S(A).

4.2.3. A *classical*[8] *prestack* is a presheaf on the opposite of the ordinary category of commutative rings.

Given a prestack S, let S_{cl} denote its *underlying classical prestack*, defined as the restriction to ordinary commutative rings. The functor $S \mapsto S_{\mathrm{cl}}$ admits a fully faithful left adjoint, embedding the category of classical prestacks as a full subcategory of prestacks.

We refer to prestacks of the form Spec(A), with A an ordinary commutative ring, as *classical affine schemes*. The functor $S \mapsto S_{\mathrm{cl}}$ sends affine schemes to classical affine schemes: we have $\mathrm{Spec}(A)_{\mathrm{cl}} = \mathrm{Spec}(H^0(A))$ for any simplicial commutative ring A.

4.3. Quasi-coherent sheaves.

4.3.1. Let $S = \mathrm{Spec}(A)$ be an affine scheme. A *quasi-coherent module* on S is the datum of an A-module. We write $\mathcal{O}_{\mathrm{Spec}(A)}$ for the quasi-coherent module given by A, viewed as a module over itself.

4.3.2. Let S be a prestack. A *quasi-coherent* $\mathcal{O}_S$*-module* consists of the following data:

(1) For every affine scheme Spec(A) and every morphism $s : \mathrm{Spec}(A) \to S$, a quasi-coherent module $\mathcal{F}_s$ on $\mathcal{O}_{\mathrm{Spec}(A)}$.

[8]The adjective *classical* refers to the fact that they are defined on non-derived objects (ordinary commutative rings, not simplicial commutative rings). In the literature they have been studied by C. Simpson and others under the name *higher prestacks*, since they may take values in arbitrary spaces, not just groupoids. In our terminology, prestacks are "higher" by default, and "non-higher" prestacks are 1-*truncated* prestacks.

(2) For every pair of morphisms $s : \mathrm{Spec}(\mathrm{A}) \to \mathrm{S}$, $s' : \mathrm{Spec}(\mathrm{B}) \to \mathrm{S}$ fitting into a commutative triangle

$$\begin{tikzcd} \mathrm{Spec}(\mathrm{A}) \arrow[r, "s"] \arrow[d, "f"] & \mathrm{S}, \\ \mathrm{Spec}(\mathrm{B}) \arrow[ur, "s'"'] & \end{tikzcd}$$

an isomorphism $f^*(\mathcal{F}_{s'}) \xrightarrow{\sim} \mathcal{F}_s$.

(3) A homotopy coherent system of compatibilities between all such isomorphisms.

4.3.3. More precisely, we define the category $\mathcal{Q}coh(\mathrm{S})$ as the limit

$$\mathcal{Q}coh(\mathrm{S}) := \varprojlim_{\mathrm{Spec}(\mathrm{A}) \to \mathrm{S}} \mathcal{Q}coh(\mathrm{Spec}(\mathrm{A}))$$

in the category of arenas.

This category is *stable*, as the property of stability is stable under limits of $(\infty, 1)$-categories.

Better yet, we can define a presheaf of symmetric monoidal arenas $\mathrm{S} \mapsto \mathcal{Q}coh(\mathrm{S})$ as the right Kan extension of the presheaf $\mathrm{A} \mapsto \mathrm{A}$-*mod* along the Yoneda embedding $(\mathcal{SCR}ing)^{\mathrm{op}} \to \mathcal{P}restk$.

In particular, for each morphism of prestacks f, we have a symmetric monoidal colimit-preserving functor f^*, the inverse image functor, and its right adjoint f_*, the direct image functor.

4.3.4. Let S be a prestack. We write $\mathcal{O}_{\mathrm{S}}$ for the quasi-coherent module defined by $\mathcal{O}_{\mathrm{S},s} = \mathcal{O}_{\mathrm{Spec}(\mathrm{A})}$ for each affine scheme Spec(A) and each A-point $s : \mathrm{Spec}(\mathrm{A}) \to \mathrm{S}$. This is the unit of the symmetric monoidal structure.

Given a quasi-coherent module $\mathcal{F}$ on S, we write $\Gamma(\mathrm{X}, \mathcal{F})$ for the space of sections over an S-scheme X. This is by definition the mapping space $\mathrm{Maps}(\mathcal{O}_{\mathrm{X}}, p^*(\mathcal{F}))$, where $p : \mathrm{X} \to \mathrm{S}$ is the structural morphism.

4.4. Stacks.

4.4.1. Let $f : \mathrm{T} \to \mathrm{S}$ be a morphism of affine schemes, $\mathrm{S} = \mathrm{Spec}(\mathrm{A})$, $\mathrm{T} = \mathrm{Spec}(\mathrm{B})$.

Definition 4.4.2. *(i) The morphism f is* of finite presentation *if* B *is compact in the category of* A-*algebras.*

(ii) The morphism f is flat *if the functor $f^* : \mathcal{Q}coh(\mathrm{S}) \to \mathcal{Q}coh(\mathrm{T})$ is exact. Equivalently, the morphism $f_{\mathrm{cl}} : \mathrm{Spec}(\mathrm{H}^0(\mathrm{B})) \to \mathrm{Spec}(\mathrm{H}^0(\mathrm{A}))$ of underlying classical affine schemes is flat (i.e. $\mathrm{H}^0(\mathrm{B})$ is flat, in the usual sense, as an $\mathrm{H}^0(\mathrm{A})$-module), and the canonical morphism $\mathrm{H}^{-i}(\mathrm{A}) \otimes_{\mathrm{H}^0(\mathrm{A})} \mathrm{H}^0(\mathrm{B}) \to \mathrm{H}^{-i}(\mathrm{B})$ is invertible for each i.*

(iii) The morphism f is an open immersion *if it is a flat monomorphism*[9] *of finite presentation. Equivalently, it is flat and the morphism $f_{\mathrm{cl}} : \mathrm{Spec}(\mathrm{H}^0(\mathrm{B})) \to \mathrm{Spec}(\mathrm{H}^0(\mathrm{A}))$ of underlying classical schemes is an open immersion (in the classical sense).*

4.4.3. The Zariski topology on $\mathcal{S}ch_{\mathrm{aff}}$ is the Grothendieck topology associated to the following pretopology. A family of morphisms of affine schemes $(j_\alpha : \mathrm{U}_\alpha \to \mathrm{X})_{\alpha \in \Lambda}$ is *Zariski covering* if and only if each j_α is an open immersion, and the family of functors $(j_\alpha)^* : \mathcal{Q}coh(\mathrm{X}) \to \mathcal{Q}coh(\mathrm{U}_\alpha)$ is conservative.

Definition 4.4.4. *A* stack *is a prestack satisfying descent with respect to the Zariski topology.*

[9] I.e. the canonical morphism $\mathrm{T} \to \mathrm{T} \times_{\mathrm{S}} \mathrm{T}$ is invertible.

We write $\mathcal{S}tk$ for the full subcategory of $\mathcal{P}restk$ spanned by stacks.

4.5. Schemes.

4.5.1. Let $j : \mathrm{U} \to \mathrm{S}$ be a morphism of stacks.

Definition 4.5.2. *(i) If* S *is affine, then j is an* open immersion *if it is a monomorphism, and there exists a family $(j_\alpha : \mathrm{U}_\alpha \to \mathrm{S})_\alpha$, with each j_α an open immersion of affine schemes [Chap. 0, Definition 4.4.2] that factors through* U *and induces an effective epimorphism $\sqcup_\alpha \mathrm{U}_\alpha \to \mathrm{U}$.*

(ii) For general S*, the morphism j is an* open immersion *if for each simplicial commutative ring* A *and each* A*-point $s : \mathrm{Spec(A)} \to \mathrm{S}$, the base change $\mathrm{U} \times_{\mathrm{S}} \mathrm{Spec(A)} \to \mathrm{Spec(A)}$ is an open immersion in the sense of (i).*

4.5.3. Let S be a stack. A *Zariski cover* of S is a small family of open immersions of stacks $(j_\alpha : \mathrm{U}_\alpha \to \mathrm{S})_\alpha$ such that the canonical morphism $\sqcup_\alpha \mathrm{U}_\alpha \to \mathrm{S}$ is surjective (i.e. an effective epimorphism in the topos of stacks). If each U_α is an affine scheme, we call this an *affine Zariski cover.*

We define:

Definition 4.5.4. *A* scheme *is a stack* S *which admits an affine Zariski cover.*

We write $\mathcal{S}ch$ for the full subcategory of $\mathcal{S}tk$ spanned by schemes. It is closed under coproducts and fibred products.

Definition 4.5.5. *(i) A scheme* S *is* quasi-compact *if for any Zariski cover $(j_\alpha : \mathrm{U}_\alpha \to \mathrm{S})_{\alpha \in \Lambda}$, there exists a finite subset $\Lambda_0 \subset \Lambda$ such that the family $(j_\alpha)_{\alpha \in \Lambda_0}$ is still a Zariski cover.*

(ii) A morphism of schemes $f : \mathrm{T} \to \mathrm{S}$ is quasi-compact *if for any simplicial commutative ring* A *and any* A*-point $s : \mathrm{Spec(A)} \to \mathrm{S}$, the scheme $\mathrm{T} \times_{\mathrm{S}} \mathrm{Spec(A)}$ is quasi-compact.*

(iii) A scheme S *is* quasi-separated *if for any open immersions $\mathrm{U} \hookrightarrow \mathrm{S}$ and $\mathrm{V} \hookrightarrow \mathrm{S}$, with* U *and* V *affine, the intersection $\mathrm{U} \times_{\mathrm{S}} \mathrm{V}$ is quasi-compact.*

4.5.6. We define a *classical scheme* to be a Zariski sheaf of *sets* on the category $(\mathcal{CR}ing)^{\mathrm{op}}$, admitting a Zariski affine cover. This is equivalent to the definition of scheme given in [EGA $\mathrm{I_S}$].

Given a scheme S, the underlying classical prestack $\mathrm{S_{cl}}$ takes values in sets, and is a classical scheme. We therefore refer to $\mathrm{S_{cl}}$ as the *underlying classical scheme* of S.

4.6. Closed immersions.

4.6.1. Let $f : \mathrm{Y} \to \mathrm{X}$ be a morphism of schemes. We say that the morphism f is *affine* if, for any simplicial commutative ring A and A-point $x : \mathrm{Spec(A)} \to \mathrm{X}$, the base change $\mathrm{Y} \times_{\mathrm{X}} \mathrm{Spec(A)}$ is an affine scheme.

4.6.2. If $\mathrm{X} = \mathrm{Spec(A)}$ and $\mathrm{Y} = \mathrm{Spec(B)}$ are affine, the morphism f is a *closed immersion* if the homomorphism $\mathrm{A} \to \mathrm{B}$ induces a surjection $\mathrm{H}^0(\mathrm{A}) \to \mathrm{H}^0(\mathrm{B})$.

In general the morphism f is a *closed immersion* if it is affine, and for any simplicial commutative ring A and A-point $x : \mathrm{Spec(A)} \to \mathrm{X}$, the base change $\mathrm{Y} \times_{\mathrm{X}} \mathrm{Spec(A)} \to \mathrm{Spec(A)}$ is a closed immersion of affine schemes.

Equivalently, f is a closed immersion if and only if it induces a closed immersion on underlying classical schemes.

4.6.3. Let $i : \mathrm{Z} \hookrightarrow \mathrm{S}$ be a closed immersion of schemes. Let U be the prestack defined as follows: for a simplicial commutative ring A, its A-points are A-points $s : \mathrm{Spec(A)} \to \mathrm{S}$ such that the base change $\mathrm{Spec(A)} \times_{\mathrm{S}} \mathrm{Z}$ is the empty scheme. One can show that U is a scheme, and that the canonical morphism $\mathrm{U} \to \mathrm{S}$ is an open immersion.

We call $j : \mathrm{U} \hookrightarrow \mathrm{S}$ the *complementary open immersion* to i.

4.7. Vector bundles.

4.7.1. Just as in [Chap. 0, Paragraph 4.3], we can define a notion of *quasi-coherent algebra* on a prestack S, such that the category of quasi-coherent algebras on Spec(A) coincides with the category of A-algebras.

4.7.2. Let S be a scheme and $\mathcal{A}$ a quasi-coherent algebra on S. Consider the presheaf $\mathrm{Spec}_{\mathrm{S}}(\mathcal{A})$ on the category of schemes over S, which sends an S-scheme X with structural morphism f to the space of quasi-coherent algebra homomorphisms $\mathrm{Maps}(f^*(\mathcal{A}), \mathcal{O}_{\mathrm{X}})$.

The presheaf $\mathrm{Spec}_{\mathrm{S}}(\mathcal{A})$ clearly satisfies Zariski descent. Hence it defines a stack over S (there is a canonical equivalence $\mathcal{S}h(\mathcal{S}ch_{/\mathrm{S}}) = \mathcal{S}h(\mathcal{S}ch)_{/\mathrm{S}} = \mathcal{S}h(\mathcal{S}ch_{\mathrm{aff}})_{/\mathrm{S}})$, which we call the *relative spectrum* of the quasi-coherent algebra $\mathcal{A}$.

In fact, we have:

Lemma 4.7.3. *Let $\mathcal{A}$ be a* connective *quasi-coherent algebra over a scheme* S*. Then the stack* $\mathrm{Spec}_{\mathrm{S}}(\mathcal{A})$ *is a scheme.*

This follows from functoriality in S, and the fact that for $\mathrm{S} = \mathrm{Spec(A)}$ affine, we have $\mathrm{Spec}_{\mathrm{S}}(\mathcal{A}) = \mathrm{Spec}(\Gamma(\mathrm{S}, \mathcal{A}))$.

4.7.4. Let $\mathcal{F}$ be a connective quasi-coherent module on S. The (connective) quasi-coherent algebra $\mathrm{Sym}_{\mathcal{O}_{\mathrm{S}}}(\mathcal{F})$ is defined by $\mathrm{Sym}_{\mathcal{O}_{\mathrm{S}}}(\mathcal{F})_s := \mathrm{Sym}_{\mathcal{O}_{\mathrm{S},s}}(\mathcal{F}_s)$ for each simplicial commutative ring A and A-point $s : \mathrm{Spec(A)} \to \mathrm{S}$.

4.7.5. A connective quasi-coherent module $\mathcal{F}$ on S is *locally free* of rank n if there exists a Zariski cover $(j_\alpha : \mathrm{U}_\alpha \to \mathrm{S})_\alpha$ such that each inverse image $j_\alpha^*(\mathcal{F})$ is a free quasi-coherent $\mathcal{O}_{\mathrm{S}_\alpha}$-module of rank n, i.e. $j_\alpha^*(\mathcal{F}) = \mathcal{O}_{\mathrm{S}_\alpha}^{\oplus n}$.

Given a locally free module of finite rank $\mathcal{F}$, we define:

Definition 4.7.6. *The* vector bundle *associated to $\mathcal{F}$ is the* S*-scheme* $\mathbf{V}(\mathcal{F}) := \mathrm{Spec}_{\mathrm{S}}(\mathrm{Sym}_{\mathcal{O}_{\mathrm{S}}}(\mathcal{F}^\vee))$.

Note that any global section $s \in \Gamma(\mathrm{S}, \mathcal{F})$ defines a section $s : \mathrm{S} \hookrightarrow \mathbf{V}(\mathcal{F})$ of the structural morphism, which is a closed immersion. In particular, any vector bundle admits a zero section.

4.7.7. For an integer $n \geqslant 0$, we define the *affine space* of dimension n over a scheme S, denoted $\mathbf{A}^n_{\mathrm{S}}$, to be the total space of the free $\mathcal{O}_{\mathrm{S}}$-module $\mathcal{O}_{\mathrm{S}}^{\oplus n}$:

$$\mathbf{A}^n_{\mathrm{S}} = \mathbf{V}(\mathcal{O}_{\mathrm{S}}^{\oplus n}).$$

For any morphism of schemes $f : \mathrm{T} \to \mathrm{S}$, we have $\mathbf{A}^n_{\mathrm{S}} \times_{\mathrm{S}} \mathrm{T} = \mathbf{A}^n_{\mathrm{T}}$. Since the structural morphism $\mathbf{A}^n_{\mathrm{S}} \to \mathrm{S}$ is flat, it follows that there is a canonical isomorphism $(\mathbf{A}^n_{\mathrm{S}})_{\mathrm{cl}} = \mathbf{A}^n_{\mathrm{S}_{\mathrm{cl}}}$.

4.7.8. The *affine line* $\mathbf{A}^1_{\mathrm{S}}$ over S is the affine space of dimension 1. Since the quasi-coherent module $\mathcal{O}_{\mathrm{S}}$ has a unit section (being a quasi-coherent *algebra*), the affine line admits both a zero and a unit section.

5. Local properties of morphisms

The reference for this section is [TV08] [MT10].

5.1. Square-zero extensions.

5.1.1. Let $p : \mathrm{Y} \to \mathrm{X}$ be a morphism of affine schemes, $\mathrm{X} = \mathrm{Spec}(\mathrm{A})$, $\mathrm{Y} = \mathrm{Spec}(\mathrm{B})$. Given a connective quasi-coherent module $\mathcal{F}$ on Y, we let

$$\mathrm{Y} \hookrightarrow \mathrm{Y}_{\mathcal{F}} := \mathrm{Spec}(\mathrm{B} \oplus \mathrm{M})$$

denote the *trivial square-zero extension of* Y *along* $\mathcal{F}$, where $\mathrm{M} = \Gamma(\mathrm{Y}, \mathcal{F})$.

The morphism $\mathrm{Y} \hookrightarrow \mathrm{Y}_{\mathcal{F}}$ is the closed immersion induced by the homomorphism $\mathrm{B} \oplus \mathrm{M} \to \mathrm{B}$, $(b, m) \mapsto b$.

5.1.2. A *derivation* of Y over X with values in $\mathcal{F}$, is a retraction of the morphism $\mathrm{Y} \hookrightarrow \mathrm{Y}_{\mathcal{F}}$ (in the category of affine schemes over X). There is a canonical retraction, the *trivial derivation* d_{triv}, defined by the morphism $\mathrm{B} \to \mathrm{B} \oplus \mathrm{M}$, $b \mapsto (b, 0)$.

Let $\mathrm{Der}(\mathrm{Y}/\mathrm{X}, \mathcal{F})$ denote the space of derivations in $\mathcal{F}$.

5.1.3. Let $\mathcal{F}$ be a 0-*connected* quasi-coherent module on Y.

Any derivation d of Y/X valued in $\mathcal{F}$ gives rise to a *square-zero extension* $i : \mathrm{Y} \hookrightarrow \mathrm{Y}_d$. This is the closed immersion (in fact, nil-immersion[10]) defined as the cobase change of the trivial derivation along d, so that there is a cocartesian square

$$\begin{tikzcd} \mathrm{Y}_{\mathcal{F}} \arrow[r, "d_{\mathrm{triv}}"] \arrow[d, "d"] & \mathrm{Y} \arrow[d] \\ \mathrm{Y} \arrow[r, "i"] & \mathrm{Y}_d \end{tikzcd} \tag{5.1}$$

in the category of affine schemes.

5.1.4. The following important fact reduces many proofs in derived algebraic geometry to their classical analogues, by induction along square-zero extensions.

Proposition 5.1.5. *Let* $\mathrm{S} = \mathrm{Spec}(\mathrm{A})$ *be an affine scheme. Then there exists a sequence of nil-immersions of affine schemes*

$$\mathrm{S}_{\mathrm{cl}} = \mathrm{S}^{\geqslant 0} \hookrightarrow \mathrm{S}^{\geqslant -1} \hookrightarrow \cdots \hookrightarrow \mathrm{S}^{\geqslant -n} \hookrightarrow \cdots \hookrightarrow \mathrm{S}, \tag{5.2}$$

with $\mathrm{S}^{\geqslant -n} = \mathrm{Spec}(\mathrm{A}^{\geqslant -n})$*, satisfying the following properties:*

(i) For each $n \geqslant 0$*, the homomorphism* $\mathrm{A} \to \mathrm{A}^{\geqslant -n}$ *identifies* $\mathrm{A}^{\geqslant -n}$ *as the* n*-truncation of the simplicial commutative ring* A.

(ii) The sequence is functorial in A.

(iii) The canonical morphism $\mathrm{A} \to \varprojlim_{n \geqslant 0} \mathrm{A}^{\geqslant -n}$ *is invertible.*

(iv) Each morphism $\mathrm{S}^{\geqslant -n} \hookrightarrow \mathrm{S}^{\geqslant -n-1}$ $(n \geqslant 0)$ *is a square-zero extension by a derivation valued in* $\mathrm{H}^{-n}(\mathcal{O}_{\mathrm{S}})[n+1]$.

Further, this sequence is uniquely characterized, up to isomorphism of diagrams indexed on the poset of nonnegative integers, by the property (i).

The sequence (5.2) is often called the *Postnikov tower* of A.

[10] See [Chap. 0, Paragraph 5.5].

5.2. The cotangent sheaf.

5.2.1. Let $p : \mathrm{Y} \to \mathrm{X}$ be a morphism of affine schemes. We have:

Proposition 5.2.2. *The functor $\mathcal{F} \mapsto \mathrm{Der}(\mathrm{Y}/\mathrm{X}, \mathcal{F})$ is representable by a connective quasi-coherent module $\mathcal{T}^*_{\mathrm{Y}/\mathrm{X}}$ on* Y*.*

The quasi-coherent module $\mathcal{T}^*_{\mathrm{Y}/\mathrm{X}}$ on Y is called the (relative) *cotangent sheaf* of the morphism $p : \mathrm{Y} \to \mathrm{X}$. We obtain the absolute cotangent sheaf $\mathcal{T}^*_{\mathrm{S}}$ by taking the relative cotangent sheaf of the morphism $\mathrm{S} \to \mathrm{Spec}(\mathbf{Z})$.

5.2.3. The following fact is crucial.

Lemma 5.2.4. *Let $\mathrm{Z} \xrightarrow{g} \mathrm{Y} \xrightarrow{f} \mathrm{X}$ be a sequence of morphisms of affine schemes. Then there is a canonical exact triangle*

$$g^*(\mathcal{T}^*_{\mathrm{Y}/\mathrm{X}}) \to \mathcal{T}^*_{\mathrm{Z}/\mathrm{X}} \to \mathcal{T}^*_{\mathrm{Z}/\mathrm{Y}} \tag{5.3}$$

of quasi-coherent sheaves on Z*.*

In particular, we see that the relative cotangent sheaf $\mathcal{T}^*_{\mathrm{Y}/\mathrm{X}}$ is the cofibre of the canonical morphism $f^*(\mathcal{T}^*_{\mathrm{X}}) \to \mathcal{T}^*_{\mathrm{Y}}$.

5.2.5. The cotangent sheaf of a vector bundle has a particularly simple description:

Lemma 5.2.6. *Let* S *be an affine scheme. For any connective quasi-coherent module $\mathcal{F}$ on* S*, we have a canonical isomorphism*

$$\mathcal{T}^*_{\mathrm{Spec}_{\mathrm{S}}(\mathrm{Sym}_{\mathcal{O}_{\mathrm{S}}}(\mathcal{F}))/\mathrm{S}} \xrightarrow{\sim} p^*(\mathcal{F}),$$

where p denotes the structural morphism of the S*-scheme* $\mathrm{Spec}_{\mathrm{S}}(\mathrm{Sym}_{\mathcal{O}_{\mathrm{S}}}(\mathcal{F}))$*.*

5.2.7. Let $f : \mathrm{Y} \to \mathrm{X}$ be a morphism of affine schemes, A a simplicial commutative ring, and $y : \mathrm{Spec}(\mathrm{A}) \to \mathrm{Y}$ an A-point. Let $f(y)$ denote the induced A-point $\mathrm{Spec}(\mathrm{A}) \to \mathrm{Y} \to \mathrm{X}$.

The *differential* of f at y is the canonical morphism of quasi-coherent sheaves on Spec(A)

$$df_y : \mathcal{T}^*_{\mathrm{X},f(y)} \to \mathcal{T}^*_{\mathrm{Y},y}. \tag{5.4}$$

5.2.8. Let S be an affine scheme and let $f : \mathrm{S} \to \mathbf{A}^1$ be a morphism, defining a global section $f \in \Gamma(\mathrm{S}, \mathcal{O}_{\mathrm{S}})$.

For any simplicial commutative ring A and A-point $s : \mathrm{Spec}(\mathrm{A}) \to \mathrm{S}$, the differential of f at s defines (by [Chap. 0, Lemma 5.2.6]) a point df_s of (the underlying space of) $\mathcal{T}^*_{\mathrm{X},x}$.

5.2.9. Let $p : \mathrm{Y} \to \mathrm{X}$ be a morphism of *schemes*. For any simplicial commutative ring A, A-point $y : \mathrm{Spec}(\mathrm{A}) \to \mathrm{Y}$, and 0-connected quasi-coherent module $\mathcal{F}$ on Y, a *derivation at y of p with values in $\mathcal{F}$* is a commutative triangle

$$\begin{tikzcd} & \mathrm{Spec}(\mathrm{A}) \arrow[dl, hook] \arrow[dr, "y"] & \\ \mathrm{Spec}(\mathrm{A})_{\mathcal{F}} \arrow[rr, "d"] & & \mathrm{Y} \end{tikzcd}$$

in the category of X-schemes.

We write $\mathrm{Der}_y(\mathrm{Y}/\mathrm{X}, \mathcal{F})$ for the space of derivations at y.

5.2.10. The functor $\mathcal{F} \mapsto \mathrm{Der}_y(\mathrm{Y}/\mathrm{X}, \mathcal{F})$ is represented by a connective quasi-coherent module $\mathcal{T}^*_{\mathrm{Y}/\mathrm{X},y}$ on $\mathrm{Spec}(\mathrm{A})$, called the relative cotangent sheaf of p at y:

$$\mathrm{Der}_y(\mathrm{Y}/\mathrm{X}, \mathcal{F}) \xrightarrow{\sim} \mathrm{Maps}_{\mathcal{Q}coh(\mathrm{Spec}(\mathrm{A}))}(\mathcal{T}^*_{\mathrm{Y}/\mathrm{X},y}, \mathcal{F}).$$

5.2.11. Given a simplicial commutative ring A and an A-point $y : \mathrm{Spec}(\mathrm{A}) \to \mathrm{Y}$, a simplicial commutative ring B and a B-point $y' : \mathrm{Spec}(\mathrm{B}) \to \mathrm{Y}$, and a morphism of affine schemes $f : \mathrm{Spec}(\mathrm{B}) \to \mathrm{Spec}(\mathrm{A})$ such that $y \circ f = y'$, we obtain a canonical morphism of quasi-coherent modules on Spec(B)

$$f^*(\mathcal{T}^*_{\mathrm{Y}/\mathrm{X},y}) \to \mathcal{T}^*_{\mathrm{Y}/\mathrm{X},y'}$$

which is *invertible.*

Moreover, the data of the quasi-coherent modules $\mathcal{T}^*_{\mathrm{Y}/\mathrm{X},y}$, as y varies over A-points of Y (with A an arbitrary simplicial commutative ring), together with the above isomorphisms, is compatible in a homotopy coherent way, and can therefore be refined to a connective quasi-coherent module $\mathcal{T}^*_{\mathrm{Y}/\mathrm{X}}$ defined on the scheme Y.

5.3. Smooth and étale morphisms.

5.3.1. Let $p : \mathrm{Y} \to \mathrm{X}$ be a morphism of affine schemes.

Definition 5.3.2. *(i) The morphism p is étale if it is of finite presentation and the cotangent sheaf $\mathcal{T}^*_{\mathrm{Y}/\mathrm{X}}$ is zero. Equivalently, p is flat and the underlying morphism of classical schemes $p_{\mathrm{cl}} : \mathrm{Y}_{\mathrm{cl}} \to \mathrm{X}_{\mathrm{cl}}$ is étale in the sense of [EGA IV$_4$].*

(ii) The morphism p is smooth *if it is of finite presentation and the complex sheaf $\mathcal{T}^*_{\mathrm{Y}/\mathrm{X}}$ is locally free of finite rank. Equivalently, p is flat and the underlying morphism of classical schemes $p_{\mathrm{cl}} : \mathrm{Y}_{\mathrm{cl}} \to \mathrm{X}_{\mathrm{cl}}$ is smooth in the sense of [EGA IV$_4$].*

In general, for a morphism of schemes $p : \mathrm{Y} \to \mathrm{X}$, we define étaleness and smoothness Zariski-locally on the source. That is:

Definition 5.3.3. *The morphism p is étale (resp. smooth, flat, locally of finite presentation) if there exists affine Zariski covers $(\mathrm{Y}_\alpha \hookrightarrow \mathrm{Y})_\alpha$ and $(\mathrm{X}_\beta \hookrightarrow \mathrm{X})_\beta$ together with the data of, for each α, an index β and a morphism of affine schemes $\mathrm{Y}_\alpha \hookrightarrow \mathrm{X}_\beta$ which is étale (resp. smooth, flat, locally of finite presentation) and fits in a commutative square*

$$\begin{tikzcd} \mathrm{Y}_\alpha \arrow[r] \arrow[d, hook] & \mathrm{X}_\beta \arrow[d, hook] \\ \mathrm{Y} \arrow[r] & \mathrm{X}. \end{tikzcd}$$

Note that open immersions are étale, and étale morphisms are smooth.

By [Chap. 0, Lemma 5.2.6], any vector bundle $\mathrm{E} \to \mathrm{S}$ is smooth. In particular, affine spaces (and hence also projective spaces) are smooth.

5.3.4. The following, a derived version of [EGA IV$_4$, Thm. 17.11.4], is standard:

Proposition 5.3.5. *A morphism $p : \mathrm{Y} \to \mathrm{X}$ is smooth if and only if, Zariski-locally on Y, there exists a factorization of p as a composite*

$$\mathrm{Y} \xrightarrow{q} \mathbf{A}^n \times \mathrm{X} \xrightarrow{r} \mathrm{X} \tag{5.5}$$

for some integer $n \geqslant 0$, where q is étale and r is the canonical projection.

Proof. Since $p_{\mathrm{cl}} : \mathrm{Y}_{\mathrm{cl}} \to \mathrm{X}_{\mathrm{cl}}$ is smooth, we can find, Zariski-locally on Y_{cl}, sections $(f_i)_{i=1,\ldots,n}$ of $\mathcal{O}_{\mathrm{Y}_{\mathrm{cl}}}$ whose differentials $d_{\mathrm{Y}_{\mathrm{cl}}/\mathrm{X}_{\mathrm{cl}}}(f_i)$ form a basis of the locally free sheaf $\mathcal{T}^*_{\mathrm{Y}_{\mathrm{cl}}/\mathrm{X}_{\mathrm{cl}}}$ (see the proof of [EGA IV_4, Thm. 17.11.4]). Choosing lifts of these sections to $\mathcal{O}_{\mathrm{Y}}$, we obtain a morphism $q : \mathrm{Y} \to \mathbf{A}^n \times \mathrm{X}$ such that the morphism

$$q^*(\mathcal{T}^*_{\mathbf{A}^n \times \mathrm{X}/\mathrm{X}}) \to \mathcal{T}^*_{\mathrm{Y}/\mathrm{X}}$$

is invertible, and hence q is étale by the exact triangle (5.3). □

5.4. Deformation along square-zero extensions.

5.4.1. Let S be an affine scheme and S′ a square-zero extension of S by a derivation $d : \mathcal{T}^*_{\mathrm{S}} \to \mathcal{F}$, for some 0-connected quasi-coherent module $\mathcal{F}$. Let X be an affine scheme over S with structural morphism p.

Definition 5.4.2. *A* deformation *of* X *along the square-zero extension* $\mathrm{S} \hookrightarrow \mathrm{S}'$ *is an affine scheme* X′ *over* S′ *together with an isomorphism* $\mathrm{X} \to \mathrm{X}' \times_{\mathrm{S}'} \mathrm{S}$.

In other words, a deformation of X is a cartesian square:

$$\begin{tikzcd} \mathrm{X} \arrow[r, hook, dashed] \arrow[d] & \mathrm{X}' \arrow[d, dashed] \\ \mathrm{S} \arrow[r, hook] & \mathrm{S}' \end{tikzcd}$$

5.4.3. We have:

Lemma 5.4.4. *The datum of a deformation of* X *along* $\mathrm{S} \hookrightarrow \mathrm{S}'$ *is equivalent to the datum of a null-homotopy of the composite*

$$\mathcal{T}^*_{\mathrm{X}/\mathrm{S}}[-1] \to p^*(\mathcal{T}^*_{\mathrm{S}}) \to p^*(\mathcal{F}).$$

Given such a null-homotopy, one obtains a derivation $d' : \mathcal{T}^*_{\mathrm{X}} \to p^*(\mathcal{F})$; the deformation X′ is constructed as the square-zero extension of X along d'.

5.4.5. For example, if p is *smooth*, then any morphism $\mathcal{T}^*_{\mathrm{X}/\mathrm{S}}[-1] \to p^*(\mathcal{F})$ must be null-homotopic; hence X admits a deformation along any square-zero extension $\mathrm{S} \hookrightarrow \mathrm{S}'$. If p is further *étale*, then this deformation is unique.

5.5. Cobase change along nil-immersions.

5.5.1. A *nil-immersion* is a closed immersion $i : \mathrm{X}_0 \hookrightarrow \mathrm{X}$ which induces an isomorphism $(\mathrm{X}_0)_{\mathrm{cl}} \to \mathrm{X}_{\mathrm{cl}}$ on underlying classical schemes.

5.5.2. Let $i : \mathrm{X}_0 \hookrightarrow \mathrm{X}$ be a nil-immersion and $f : \mathrm{X}_0 \to \mathrm{Y}_0$ a morphism of schemes.

We have:

Lemma 5.5.3. *(i) The cobase change of f along the nil-immersion i is representable in the category of schemes. That is, there exists a scheme* Y *fitting in the cocartesian square*

$$\begin{tikzcd} \mathrm{X}_0 \arrow[r, hook, "i"] \arrow[d, "f"] & \mathrm{X} \arrow[d] \\ \mathrm{Y}_0 \arrow[r] & \mathrm{Y}. \end{tikzcd}$$

(ii) If f is affine then the morphism $\mathrm{Y}_0 \to \mathrm{Y}$ *is a closed nil-immersion.*

(iii) If f is an open immersion, then so is the morphism $\mathrm{X} \to \mathrm{Y}$.

5.6. Lifting smooth morphisms along closed immersions.

5.6.1. The following is a derived version of [EGA IV_4, Prop. 18.1.1]:

Proposition 5.6.2. *Let $i : \mathrm{Z} \hookrightarrow \mathrm{S}$ be a closed immersion of schemes. For any smooth (resp. étale) morphism $p : \mathrm{X} \to \mathrm{Z}$, there exists, Zariski-locally on X, a smooth (resp. étale) morphism $q : \mathrm{Y} \to \mathrm{S}$, and a cartesian square*

$$\begin{tikzcd} \mathrm{X} \arrow[r] \arrow[d, "p"] & \mathrm{Y} \arrow[d, "q"] \\ \mathrm{Z} \arrow[r, hook] & \mathrm{S}. \end{tikzcd}$$

Proof. First we consider the étale case. The question being Zariski-local, we may assume that S, Z and X are affine. Consider the Postnikov towers ([Chap. 0, Proposition 5.1.5])

$$\mathrm{S}_{\mathrm{cl}} = \mathrm{S}^{\geqslant 0} \hookrightarrow \mathrm{S}^{\geqslant -1} \hookrightarrow \cdots \hookrightarrow \mathrm{S}^{\geqslant -n} \hookrightarrow \cdots \hookrightarrow \mathrm{S}$$
$$\mathrm{Z}_{\mathrm{cl}} = \mathrm{Z}^{\geqslant 0} \hookrightarrow \mathrm{Z}^{\geqslant -1} \hookrightarrow \cdots \hookrightarrow \mathrm{Z}^{\geqslant -n} \hookrightarrow \cdots \hookrightarrow \mathrm{Z}$$

for S and Z, respectively. Since p is flat, the Postnikov tower for X is identified with the base change of the Postnikov tower of Z.

For a fixed integer $n \geqslant 0$, consider the following claim:

$(*)$ There exists, Zariski-locally on $\mathrm{X}^{\geqslant -n}$, an étale morphism $q^{\geqslant -n} : \mathrm{Y}^{\geqslant -n} \to \mathrm{S}^{\geqslant -n}$ and a cartesian square

$$\begin{tikzcd} \mathrm{X}^{\geqslant -n} \arrow[r, hook] \arrow[d, "p^{\geqslant -n}"] & \mathrm{Y}^{\geqslant -n} \arrow[d, "q^{\geqslant -n}"] \\ \mathrm{Z}^{\geqslant -n} \arrow[r, hook] & \mathrm{S}^{\geqslant -n}. \end{tikzcd}$$

Note that it suffices to show that $(*)$ holds for each $n \geqslant 0$, since we can conclude by passing to filtered colimits. For $n = 0$, the claim is [EGA IV_4, Prop. 18.1.1].

We proceed by induction; assume that the claim holds for a fixed n. We define $\mathrm{Y}^{\geqslant -n-1}$ to be the deformation of $\mathrm{Y}^{\geqslant -n}$ along the square-zero extension $\mathrm{S}^{\geqslant -n} \hookrightarrow \mathrm{S}^{\geqslant -n-1}$, which exists by [Chap. 0, Lemma 5.4.4]. Note that $\mathrm{X}^{\geqslant -n-1}$ is itself a deformation of $\mathrm{X}^{\geqslant -n}$ along the square-zero extension $\mathrm{Z}^{\geqslant -n} \hookrightarrow \mathrm{Z}^{\geqslant -n-1}$. That the resulting square is cartesian is a straightforward verification.

For the smooth case, the claim follows from the étale case and from [Chap. 0, Proposition 5.3.5]. □

6. Global properties of morphisms

The reference for this section is [GR16, Chap. II.2].

6.1. Proper morphisms.

6.1.1. Let $f : \mathrm{Y} \to \mathrm{X}$ be a morphism of schemes. We say that f is *of finite type* if the underlying morphism of classical schemes f_{cl} is of finite type in the sense of [EGA $\mathrm{I_S}$].

We say that f is *separated* if the diagonal morphism $\mathrm{Y} \to \mathrm{Y} \times_{\mathrm{X}} \mathrm{Y}$ is a closed immersion. Equivalently, the underlying morphism of classical schemes f_{cl} is separated in the sense of [EGA $\mathrm{I_S}$].

The morphism f is *proper* if the induced morphism of underlying classical schemes $f_{\mathrm{cl}} : \mathrm{Y}_{\mathrm{cl}} \to \mathrm{X}_{\mathrm{cl}}$ is proper in the sense of [EGA II].

6.1.2. Let $\mathcal{E}$ be a locally free module of finite rank over a scheme S.

Consider the presheaf on the category of S-schemes sending an S-scheme X with structural morphism f to the space of direct summands of $f^*(\mathcal{E})^\vee$ which are locally free of rank 1.

This satisfies Zariski descent and corresponds to a stack over S which we denote $\mathbf{P}(\mathcal{E})$ and call the *projective bundle* associated to $\mathcal{E}$. One can construct an affine Zariski cover of $\mathbf{P}(\mathcal{E})$, so that it is in fact a scheme.

Further, since the structural morphism $\mathbf{P}(\mathcal{E}) \to \mathrm{S}$ is flat, we have $\mathbf{P}(\mathcal{E})_{\mathrm{cl}} = \mathbf{P}(\mathcal{E}) \times_{\mathrm{S}} \mathrm{S}_{\mathrm{cl}} = \mathbf{P}(i^*\mathcal{E})$, where $i : \mathrm{S}_{\mathrm{cl}} \hookrightarrow \mathrm{S}$ is the canonical inclusion. For classical schemes, our construction coincides with that of [EGA $\mathrm{I_S}$] by definition, and in [EGA II] it is proved that $\mathbf{P}(i^*\mathcal{E}) \to \mathrm{S}_{\mathrm{cl}}$ is proper. Hence we obtain:

Lemma 6.1.3. *The stack $\mathbf{P}(\mathcal{E})$ is a scheme, and the structural morphism $p : \mathbf{P}(\mathcal{E}) \to \mathrm{S}$ is proper.*

6.1.4. Taking $\mathcal{E} = \mathcal{O}_{\mathrm{S}}^{\oplus n+1}$, we obtain the *projective space* of dimension n over S:

$$\mathbf{P}^n_{\mathrm{S}} := \mathbf{P}(\mathcal{O}_{\mathrm{S}}^{\oplus n+1})$$

for each $n \geqslant 0$.

In this case, an affine Zariski cover can be chosen of the form $(\mathrm{U}_i \hookrightarrow \mathbf{P}^n_{\mathrm{S}})_{i=0,\ldots,n}$ where for each i there is an isomorphism $\mathrm{U}_i = \mathbf{A}^n_{\mathrm{S}}$.

Also, [Chap. 0, Lemma 6.1.3] can be deduced in a less direct way from the following observations: (1) for $\mathrm{S} = \mathrm{Spec}(\mathbf{Z})$, $\mathbf{P}^n_{\mathrm{S}}$ coincides with the classical proper S-scheme constructed in [EGA $\mathrm{I_S}$]; (2) the construction is functorial, i.e. $\mathbf{P}^n_{\mathrm{S}} \times_{\mathrm{S}} \mathrm{T} = \mathbf{P}^n_{\mathrm{T}}$ for any morphism $\mathrm{T} \to \mathrm{S}$; (3) the structural morphism $\mathbf{P}^n_{\mathrm{S}} \to \mathrm{S}$ is flat, so that $(\mathbf{P}^n_{\mathrm{S}})_{\mathrm{cl}} = \mathbf{P}^n_{\mathrm{S}} \times_{\mathrm{S}} \mathrm{S}_{\mathrm{cl}} = \mathbf{P}^n_{\mathrm{S}_{\mathrm{cl}}}$.

6.1.5. In particular, the *projective line* $\mathbf{P}^1_{\mathrm{S}}$ fits into a cartesian and cocartesian square of schemes

$$\begin{tikzcd}
(\mathbf{A}^1_{\mathrm{S}})^\times \arrow[r, hook] \arrow[d, hook] & \mathbf{A}^1_{\mathrm{S}} \arrow[d, hook] \\
\mathbf{A}^1_{\mathrm{S}} \arrow[r, hook] & \mathbf{P}^1_{\mathrm{S}}
\end{tikzcd}$$

where $(\mathbf{A}^1_{\mathrm{S}})^\times$ denotes the complement of the zero section $\mathrm{S} \hookrightarrow \mathbf{A}^1$.

6.2. Closure.

6.2.1. Let $f : \mathrm{Y} \to \mathrm{X}$ be a morphism of schemes. Let $\mathcal{C}losed(f)$ denote the category of factorizations

$$\begin{tikzcd}
\mathrm{Y} \arrow[rr, "f"] \arrow[dr] & & \mathrm{X} \\
& \mathrm{Z} \arrow[ur, "i"'] &
\end{tikzcd}$$

with i a closed immersion.

We have:

Lemma 6.2.2. *The category $\mathcal{C}losed(f)$ admits an initial object.*

This initial object gives in particular a closed immersion $i : \overline{f(\mathrm{Y})} \hookrightarrow \mathrm{X}$, which is called the *closure* of f.

If f is a closed immersion, then the canonical morphism $\mathrm{Y} \to \overline{f(\mathrm{Y})}$ is invertible.

6.2.3. Let $X'' \xrightarrow{g} X' \xrightarrow{f} X$ be a sequence of morphisms of schemes. We have the following transitivity property of closure:

Lemma 6.2.4. *The canonical morphism*

$$\overline{f \circ g(X'')} \to \overline{f'(\overline{g(X'')})}$$

is invertible, where $f' : \overline{g(X'')} \to X$ *is the restriction of* f*.*

6.3. Compactifications of morphisms.

6.3.1. Recall the following definition from [SGA 4, Exp. XVII, 3.2.5].

Definition 6.3.2. *Let* $f : Y \to X$ *be a morphism of schemes. A* compactification *of* f *is a commutative triangle*

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "j"'] & & X \\ & Y' \arrow[ur, "f'"'] & \end{tikzcd}$$

where j *is an open immersion and* f' *is proper.*

A morphism of compactifications is a morphism of commutative triangles. We write $\mathcal{C}ompact(f)$ for the category of compactifications of f.

6.3.3. In the setting of classical schemes, the category of compactifications $\mathcal{C}ompact(f)$ is cofiltered [SGA 4, Exp. XVII, Prop. 3.2.6], and hence contractible, for any *separated and of finite type* morphism f between quasi-compact quasi-separated schemes.

In the derived setting, we have the following statement, demonstrated in [GR16, Book-II.2, Prop. 2.1.6]:

Proposition 6.3.4. *Let* $f : Y \to X$ *be a separated morphism of finite type between quasi-compact quasi-separated schemes. Then the category* $\mathcal{C}ompact(f)$ *is contractible.*

Proof. The proof of *loc. cit.* applies *mutatis mutandis.* We briefly recall the argument here.

First, one considers the full subcategory of *dense* compactifications, for which the open immersion $j : Y \hookrightarrow Y'$ induces an isomorphism $\overline{j(Y)} \to Y'$ on the closure. Given any compactification $(Y \xrightarrow{j} Y' \to X)$, we can form a dense compactification by replacing j with $Y \to \overline{j(Y)}$. This provides a right adjoint to the inclusion of this subcategory, so it suffices to show that this subcategory is contractible[11].

By the classical Nagata compactification theorem, as generalized to quasi-compact quasi-separated schemes by Deligne [Con07], the morphism $f_{cl} : Y_{cl} \to X_{cl}$ admits a compactification $(Y_{cl} \hookrightarrow Y'_{cl} \to X_{cl})$. We define Y' to be the cobase change of Y'_{cl} along the nil-immersion $Y_{cl} \to Y$. This exists by [Chap. 0, Lemma 5.5.3] and defines a compactification $(Y \hookrightarrow Y' \to X)$ of f. Taking the associated dense compactification, this shows that the category of dense compactifications is not empty. Hence it suffices to show that it admits binary products.

For this we take two dense compactifications $(Y \hookrightarrow Y'_1 \to X)$ and $(Y \hookrightarrow Y'_2 \to X)$. Their product is given by the dense compactification defined by the closure of Y inside the fibred product $Y'_1 \times_X Y'_2$. □

[11]Recall that an adjunction of $(\infty, 1)$-categories induces isomorphisms on underlying ∞-groupoids.

CHAPTER 1

Motivic spaces and spectra

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1. Introduction

1.1. Motivic spaces.

1.1.1. Let S be a derived scheme. A *motivic space* over S is a presheaf of spaces $\mathcal{F}$ on the category of smooth derived S-schemes, satisfying the properties of *Nisnevich descent* and $\mathbf{A}^1$*-homotopy invariance.*

1.1.2. Nisnevich descent can be formulated equivalently as the following excision property (see [Chap. 1, Proposition 2.2.6]):

Consider a *Nisnevich square*, i.e. a cartesian square of smooth derived S-schemes

$$\begin{tikzcd} \mathrm{W} \arrow[r] \arrow[d] & \mathrm{V} \arrow[d, "p"] \\ \mathrm{U} \arrow[r, "j"] & \mathrm{X} \end{tikzcd} \tag{1.1}$$

with j an open immersion and p étale, and the induced morphism $p^{-1}(\mathrm{X}-\mathrm{U}) \to \mathrm{X}-\mathrm{U}$ an isomorphism of underlying reduced classical schemes. Then the induced commutative square of spaces

$$\begin{tikzcd} \Gamma(\mathrm{X},\mathcal{F}) \arrow[r, "j^*"] \arrow[d, "p^*"] & \Gamma(\mathrm{U},\mathcal{F}) \arrow[d] \\ \Gamma(\mathrm{V},\mathcal{F}) \arrow[r] & \Gamma(\mathrm{W},\mathcal{F}) \end{tikzcd}$$

is cartesian.

1.1.3. Homotopy invariance is the condition that, for any smooth derived S-scheme X, the canonical morphism of spaces

$$\Gamma(\mathrm{X},\mathcal{F}) \to \Gamma(\mathbf{A}^1 \times \mathrm{X},\mathcal{F})$$

is invertible.

1.1.4. Let $\mathcal{H}(\mathrm{S})$ denote the category of motivic spaces over S. It is easy to see that the assignment $\mathrm{S} \mapsto \mathcal{H}(\mathrm{S})$ admits the following functorialities:

Given a morphism $f : \mathrm{T} \to \mathrm{S}$ of derived schemes, there is an inverse image functor

$$f^* : \mathcal{H}(\mathrm{S}) \to \mathcal{H}(\mathrm{T})$$

which is left adjoint to a direct image functor

$$f_* : \mathcal{H}(\mathrm{T}) \to \mathcal{H}(\mathrm{S}).$$

1.1.5. If f is smooth, there is a "bonus" operation

$$f_\sharp : \mathcal{H}(\mathrm{T}) \to \mathcal{H}(\mathrm{S}),$$

left adjoint to f^*.

This operation is compatible with the operations $(f^*, f_*, \otimes, \underline{\mathrm{Hom}})$ in the sense that it satisfies various base change and projection formulas; see [Chap. 1, Sect. 6].

1.2. Motivic spectra.

1.2.1. Choose a family of pointed motivic spaces $\mathcal{T}_{\mathrm{S}}$ over each derived scheme S, together with isomorphisms

$$f^*(\mathcal{T}_{\mathrm{S}}) \xrightarrow{\sim} \mathcal{T}_{\mathrm{T}}$$

for each morphism of derived schemes $f : \mathrm{T} \to \mathrm{S}$ (and a homotopy coherent system of compatibilities between these isomorphisms).

For example, we can take the family of motivic spaces represented by projective lines $\mathbf{P}^1_{\mathrm{S}}$, pointed at ∞. These are isomorphic to the Thom spaces $\mathrm{Th}_{\mathrm{S}}(\mathbf{A}^1_{\mathrm{S}})$ (see [Chap. 1, Sect. 8]).

1.2.2. A *motivic $\mathcal{T}$-spectrum* over S is the data of a sequence of pointed motivic spaces $(\mathcal{F}_n)_{n\geqslant 0}$ over S, and isomorphisms

$$\alpha_n : \mathcal{F}_n \xrightarrow{\sim} \Omega_{\mathcal{T}}(\mathcal{F}_{n+1})$$

for each integer $n \geqslant 0$, together with a homotopy coherent system of compatibilities between these isomorphisms. Here $\mathcal{F} \mapsto \Omega_{\mathcal{T}}(\mathcal{F})$ denotes the $\mathcal{T}$-loop space functor.

1.2.3. The category $\mathcal{SH}_{\mathcal{T}_{\mathrm{S}}}(\mathrm{S})$ of motivic $\mathcal{T}_{\mathrm{S}}$-spectra can be described as the result of formally inverting the object $\mathcal{T}_{\mathrm{S}}$ with respect to the monoidal product, in the sense of [Rob15]. In particular, it admits a canonical symmetric monoidal structure, and there is a canonical symmetric monoidal functor

$$\Sigma^\infty_{\mathrm{S}} : \mathcal{H}(\mathrm{S})_\bullet \to \mathcal{SH}_{\mathcal{T}_{\mathrm{S}}}(\mathrm{S})$$

which has the universal property of being initial in the category of symmetric monoidal functors which send $\mathcal{T}_{\mathrm{S}}$ to an invertible object.

1.2.4. Our functorialities $(f^*, f_*, f_\sharp)$ extend to the assignment $\mathrm{S} \mapsto \mathcal{SH}_{\mathcal{T}_{\mathrm{S}}}(\mathrm{S})$. Hence we have inverse image functors $f^*_{\mathcal{SH}}$ for any morphism of schemes f, which admit right adjoints $f^{\mathcal{SH}}_*$ (resp. left adjoints $f^{\mathcal{SH}}_\sharp$, when f is smooth).

1.3. The localization theorem.

1.3.1. The main result of this chapter can be stated as follows.

Theorem 1.3.2 (Localization)**.** *Let $i : \mathrm{Z} \hookrightarrow \mathrm{S}$ be a closed immersion of derived schemes with quasi-compact open complement $j : \mathrm{U} \hookrightarrow \mathrm{S}$. Then the following statements hold:*

(1) For any motivic space $\mathcal{F}$ over S*, there is a canonical cocartesian square*

$$\begin{tikzcd} j_\sharp j^*(\mathcal{F}) \arrow[r] \arrow[d] & \mathcal{F} \arrow[d] \\ j_\sharp j^*(e_{\mathrm{S}}) \arrow[r] & i_* i^*(\mathcal{F}), \end{tikzcd} \tag{1.2}$$

where e_{S} denotes the terminal motivic space over S*.*

(2) For any pointed motivic space $(\mathcal{F}, x)$ over S*, there is a canonical cofibre sequence*

$$j_\sharp j^*(\mathcal{F}, x) \to (\mathcal{F}, x) \to i_* i^*(\mathcal{F}, x). \tag{1.3}$$

(3) For any motivic $\mathcal{T}_{\mathrm{S}}$-spectrum $\mathbb{E}$, there is a canonical exact triangle

$$j_\sharp j^*(\mathbb{E}) \to \mathbb{E} \to i_* i^*(\mathbb{E}). \tag{1.4}$$

This is the combination of [Chap. 1, Theorem 7.4.3], [Chap. 1, Corollary 7.4.5], and [Chap. 1, Corollary 7.4.7].

1.3.3. Over classical schemes (noetherian of finite Krull dimension), this theorem was demonstrated by Morel–Voevodsky (see [MV99]). Our proof has the same general flavour, but we make some parts of the argument more robust, so that it survives in the derived setting.

When we take a closed immersion $i : \mathrm{Z} \hookrightarrow \mathrm{S}$ of classical schemes, we recover a proof of the localization property for classical schemes without any noetherian or finite dimensional assumptions.

The point is that over a general base, our definition is *not* equivalent to the definition of Morel–Voevodsky. We only impose descent with respect to Čech covers, instead of arbitrary hypercovers. The latter condition, called *hyperdescent*, is a priori much stronger, except over noetherian finite-dimensional schemes, where they coincide. We refer to [Lur11] and [Hoy15, Appendix C] for an explanation of this distinction.

1.3.4. Since the underlying classical scheme of a derived scheme is a closed subscheme with empty complement, an immediate consequence of this result is that the unstable and stable motivic homotopy categories of a derived scheme coincide with those of its underlying classical scheme.

Corollary 1.3.5 (Topological invariance)**.** *Let* S *be a derived scheme, and write* $i : \mathrm{S}_{\mathrm{cl}} \hookrightarrow \mathrm{S}$ *for the inclusion of the underlying classical scheme. Then the adjunctions*

$$i^*_{\mathcal{H}_\bullet} : \mathcal{H}_\bullet(\mathrm{S}) \to \mathcal{H}_\bullet(\mathrm{S}_{\mathrm{cl}}), \qquad i^{\mathcal{H}_\bullet}_* : \mathcal{H}_\bullet(\mathrm{S}_{\mathrm{cl}}) \to \mathcal{H}_\bullet(\mathrm{S}),$$
$$i^*_{\mathcal{SH}} : \mathcal{SH}_{\mathcal{T}_\mathrm{S}}(\mathrm{S}) \to \mathcal{SH}_{\mathcal{T}_\mathrm{S}}(\mathrm{S}_{\mathrm{cl}}), \qquad i^{\mathcal{SH}}_* : \mathcal{SH}_{\mathcal{T}_\mathrm{S}}(\mathrm{S}_{\mathrm{cl}}) \to \mathcal{SH}_{\mathcal{T}_\mathrm{S}}(\mathrm{S})$$

are equivalences of $(\infty, 1)$*-categories.*

1.4. The exceptional operations for closed immersions.

1.4.1. In Chapter 2, we will construct the exceptional functorialities $(f_!, f^!)$ on the categories $\mathcal{SH}(\mathrm{S})$ (together with the full formalism of six operations).

In the case of closed immersions i, the exceptional direct image $i_!$ coincides with i_* by definition. We will prove (see [Chap. 1, Corollary 7.3.3]) that this admits a right adjoint $i^!$, the exceptional inverse image functor.

1.4.2. Using the localization theorem, we will verify that these operations $(i_!, i^!)$ satisfy all the desired compatibilities with the other operations $(f^*, f_*, \otimes, \underline{\mathrm{Hom}})$, i.e. base change and projection formulas. See [Chap. 1, Paragraph 7.5], [Chap. 1, Paragraph 7.6], and [Chap. 1, Paragraph 7.7].

These will be used in a critical way in Chapter 2 to obtain the formalism of six operations.

1.5. Organization of this chapter. In Sections 1-4, we construct the categories of motivic spaces and spectra, and study their basic properties. In applications it is useful to consider $\mathcal{T}$-spectra with respect to general $\mathcal{T}$, though we will obtain the full formalism of six operations in the case $\mathcal{T} = \mathbf{P}^1$.

Section 5 deals with the operations f^* (inverse image) and f_* (direct image). In Section 6 we consider an extra operation $p_\sharp$, not included in the "six operations", associated to smooth morphisms p.

In Section 7 we construct the exceptional inverse image functor $i^!$ for closed immersions. We also state the Morel–Voevodsky localization theorem and deduce some consequences.

In Section 8 we introduce the operation of Thom (de)suspension with respect to a vector bundle E. We show that the Thom space of the affine line is $\mathbf{P}^1$.

Section 9 deals with the proof of the localization theorem.

2. Motivic spaces

2.1. Fibred spaces.

2.1.1. Let S be a derived scheme[1] and write $\mathcal{S}m_{/\mathrm{S}}$ for the category of smooth schemes (of finite presentation) over S. Throughout the text, we redefine "smooth" as "smooth of finite presentation".

A *$\mathcal{S}m$-fibred space* over S is the datum of a presheaf of spaces on the category $\mathcal{S}m_{/\mathrm{S}}$. We will abbreviate this to "fibred space" or even "space" when there is no risk of confusion.[2]

Given a space $\mathcal{F}$ over S, we will write $\Gamma(\mathrm{X}, \mathcal{F})$ for the space of sections over a smooth S-scheme X.

2.1.2. Let $\mathcal{S}pc(\mathrm{S})$ denote the category of spaces over S. Recall the following from [Chap. 0, Sect. 2]:

The category $\mathcal{S}pc(\mathrm{S})$ is a topos, hence *a fortiori* a cartesian monoidal arena[3] with the property of universality of colimits.

Every smooth S-scheme X represents a space $\mathrm{h_S(X)}$, with $\Gamma(\mathrm{Y}, \mathrm{h_S(X)}) = \mathrm{Maps_S(Y, X)}$ for each smooth S-scheme Y. The assignment $\mathrm{X} \mapsto \mathrm{h_S(X)}$ defines a fully faithful functor $\mathcal{S}m_{/\mathrm{S}} \to \mathcal{S}pc(\mathrm{S})$ (the Yoneda embedding) and induces by pre-composition a canonical equivalence

$$\mathcal{F}unct_!(\mathcal{S}pc(\mathrm{S}), \mathbf{D}) \xrightarrow{\sim} \mathcal{F}unct(\mathcal{S}m_{/\mathrm{S}}, \mathbf{D}) \tag{2.1}$$

for each cocomplete category $\mathbf{D}$. In other words the category $\mathcal{S}pc(\mathrm{S})$ is freely generated under colimits by the representable spaces.

2.1.3. We will denote by $e_\mathrm{S} := e_\mathrm{S}^{\mathcal{S}pc}$ (resp. $\varnothing_\mathrm{S} := \varnothing_\mathrm{S}^{\mathcal{S}pc}$) the terminal object (resp. initial object).

We write $\times_\mathrm{S}$ for the monoidal product on $\mathcal{S}pc(\mathrm{S})$, whose unit object $\mathbf{1}_\mathrm{S}^{\mathcal{S}pc}$ is the terminal object e_S. We denote the internal hom by $\underline{\mathrm{Hom}}_\mathrm{S} := \underline{\mathrm{Hom}}_\mathrm{S}^{\mathcal{S}pc}$.

2.2. Nisnevich descent.

2.2.1. A *Nisnevich square* over S is a cartesian square of smooth S-schemes

$$\begin{tikzcd} \mathrm{U} \times_\mathrm{X} \mathrm{V} \arrow[r, hook] \arrow[d] & \mathrm{V} \arrow[d, "p"] \\ \mathrm{U} \arrow[r, hook, "j"] & \mathrm{X} \end{tikzcd} \tag{2.2}$$

such that j is an open immersion, p is étale, and there exists a closed immersion $\mathrm{Z} \hookrightarrow \mathrm{X}$ complementary to j such that the induced morphism $p^{-1}(\mathrm{Z}) \to \mathrm{Z}$ is invertible.

The *Nisnevich topology* on the category $\mathcal{S}m_{/\mathrm{S}}$ is generated by (i) the empty family over the empty scheme $\varnothing$; and (ii) families $\{j, p\}$ over a smooth S-scheme X, where j and p form a Nisnevich square as in (2.2).

[1] As per our conventions we will omit the adjective "derived" in the sequel.

[2] The explanation for this terminology is the equivalence, provided by the Grothendieck construction ("straightening/unstraightening"), between presheaves on $\mathcal{S}m_{/\mathrm{S}}$ and cartesian fibrations in spaces over $\mathcal{S}m_{/\mathrm{S}}$.

[3] Recall that by the term *arena*, we mean locally presentable $(\infty, 1)$-category; a morphism of arenas is a functor that commutes with colimits. See [Chap. 0, Paragraph 2.6].

2.2.2. Recall from [Chap. 0, Paragraph 2.9] that a fibred space $\mathcal{F}$ satisfies Nisnevich descent, or is *Nisnevich-local*, if for each Nisnevich covering family $(f_\alpha : \mathrm{X}_\alpha \to \mathrm{X})_\alpha$, the canonical morphism of spaces

$$\Gamma(\mathrm{X}, \mathcal{F}) \to \varprojlim_{n\in\mathbf{\Delta}} \Gamma(\check{\mathrm{C}}(\mathrm{X}_\alpha/\mathrm{X})_n, \mathcal{F}) \tag{2.3}$$

is invertible, where $\check{\mathrm{C}}(\mathrm{X}_\alpha/\mathrm{X})_\bullet$ denotes the Čech nerve of the morphism $\sqcup_\alpha \mathrm{X}_\alpha \to \mathrm{X}$. Equivalently, for every Nisnevich covering sieve $\mathrm{R} \hookrightarrow \mathrm{h_S}(\mathrm{X})$, the canonical morphism

$$\Gamma(\mathrm{X}, \mathcal{F}) \to \mathrm{Maps}(\mathrm{R}, \mathcal{F}) \tag{2.4}$$

is invertible.

2.2.3. A morphism $\mathcal{F} \to \mathcal{F}'$ of spaces over S is a *Nisnevich-local equivalence* if the induced morphism

$$\mathrm{L_{Nis}}(\mathcal{F}) \to \mathrm{L_{Nis}}(\mathcal{F}')$$

is invertible, or equivalently if for every Nisnevich-local space $\mathcal{G}$, the canonical morphism of spaces

$$\mathrm{Maps}(\mathcal{G}, \mathcal{F}) \to \mathrm{Maps}(\mathcal{G}, \mathcal{F}')$$

is invertible.

We will write $\mathcal{S}pc_{\mathrm{Nis}}(\mathrm{S})$ for the topos of Nisnevich-local spaces. This is an exact localization of $\mathcal{S}pc(\mathrm{S})$, i.e. the inclusion admits an exact left adjoint

$$\mathrm{L_{Nis}} : \mathcal{S}pc(\mathrm{S}) \to \mathcal{S}pc_{\mathrm{Nis}}(\mathrm{S}).$$

2.2.4. We say that a fibred space $\mathcal{F}$ satisfies *Nisnevich excision* if (i) the space $\Gamma(\varnothing, \mathcal{F})$ is contractible; and (ii) for all smooth S-schemes X and all Nisnevich squares of the form (2.2), the commutative square of spaces

$$\begin{array}{ccc} \Gamma(\mathrm{X}, \mathcal{F}) & \longrightarrow & \Gamma(\mathrm{U}, \mathcal{F}) \\ \downarrow & & \downarrow \\ \Gamma(\mathrm{V}, \mathcal{F}) & \longrightarrow & \Gamma(\mathrm{U} \times_{\mathrm{X}} \mathrm{V}, \mathcal{F}) \end{array} \tag{2.5}$$

is cartesian.

Note that the property of Nisnevich excision is stable by filtered colimits (since filtered colimits commute with finite limits in any topos).

2.2.5. By [AHW15, Thm. 3.2.5][4], we have:

Proposition 2.2.6. *Let $\mathcal{F}$ be a fibred space over a scheme* S*. Then $\mathcal{F}$ satisfies Nisnevich descent if and only if it satisfies Nisnevich excision.*

2.2.7. The following lemma follows from [Lur09b, Prop. 5.5.8.10, (3)]:

Lemma 2.2.8. *Let $(\mathrm{X}_\alpha)_\alpha$ be a finite family of smooth* S*-schemes. Then the canonical morphism of spaces over* S

$$\bigsqcup_\alpha \mathrm{h_S}(\mathrm{X}_\alpha) \to \mathrm{h_S}(\bigsqcup_\alpha \mathrm{X}_\alpha)$$

is invertible.

By [Lur09b, Lem. 5.5.8.14] it follows that the category $\mathcal{S}pc_{\mathrm{Nis}}(\mathrm{S})$ is generated under sifted colimits by the representables. In fact, we can say even more:

[4]The statement is in the language of model categories, but the proof works *mutatis mutandis* in the setting of $(\infty, 1)$-categories.

Proposition 2.2.9. *Let* S *be a scheme. Then the category* $\mathcal{S}pc_{\mathrm{Nis}}(\mathrm{S})$ *is generated under sifted colimits by the representable spaces* $\mathrm{h}_{\mathrm{S}}(\mathrm{X})$*, where* X *is a smooth* S*-scheme which is* affine[5].

Proof. Let $\mathrm{h}_{\mathrm{S}}(\mathrm{X})$ be a representable space over S. Since $\mathrm{h}_{\mathrm{S}}(\mathrm{X})$ satisfies Nisnevich descent, we can assume X is separated over S, by choosing an affine Zariski cover of X where the pairwise intersections are separated. Then we repeat the same argument to assume X is affine, by choosing an affine cover where the pairwise intersections are affine. □

2.3. $\mathbf{A}^1$-homotopy invariance.

2.3.1. A space $\mathcal{F}$ over S is $\mathbf{A}^1$*-homotopy invariant* if for every smooth S-scheme X, the canonical morphism of spaces

$$\Gamma(\mathrm{X}, \mathcal{F}) \to \Gamma(\mathbf{A}^1 \times \mathrm{X}, \mathcal{F})$$

is invertible. Here $\mathbf{A}^1$ denotes the affine line $\mathrm{Spec}(\mathbf{Z}[t])$ as usual.

Let $\mathcal{S}pc_{\mathbf{A}^1}(\mathrm{S})$ denote the full subcategory of $\mathcal{S}pc(\mathrm{S})$ spanned by $\mathbf{A}^1$-homotopy invariant spaces over S. This is an accessible localization at a small set of morphisms, and in particular the inclusion admits a left adjoint

$$\mathrm{L}_{\mathbf{A}^1} : \mathcal{S}pc(\mathrm{S}) \to \mathcal{S}pc_{\mathbf{A}^1}(\mathrm{S}) \tag{2.6}$$

called the $\mathbf{A}^1$-localization functor.

Note that the property of $\mathbf{A}^1$-homotopy invariance is stable by small colimits, since colimits of presheaves are computed sectionwise. In particular, the inclusion $\mathcal{S}pc_{\mathbf{A}^1}(\mathrm{S}) \hookrightarrow \mathcal{S}pc(\mathrm{S})$ also admits a right adjoint.

2.3.2. A morphism $\mathcal{F} \to \mathcal{F}'$ of spaces over S is an $\mathbf{A}^1$*-homotopy equivalence* if the induced morphism

$$\mathrm{L}_{\mathbf{A}^1}(\mathcal{F}) \to \mathrm{L}_{\mathbf{A}^1}(\mathcal{F}')$$

is invertible, or equivalently if for every $\mathbf{A}^1$-homotopy invariant space $\mathcal{G}$, the canonical morphism of spaces

$$\mathrm{Maps}(\mathcal{G}, \mathcal{F}) \to \mathrm{Maps}(\mathcal{G}, \mathcal{F}')$$

is invertible.

Let i_0 (resp. i_1) denote the zero section (resp. unit section) $\mathrm{S} \hookrightarrow \mathbf{A}^1 \times \mathrm{S}$. Given two morphisms $f, g : \mathcal{F} \rightrightarrows \mathcal{G}$ of spaces over S, an *elementary* $\mathbf{A}^1$*-homotopy* from f to g is a morphism

$$\varphi : \mathrm{h}_{\mathrm{S}}(\mathbf{A}^1 \times \mathrm{S}) \times \mathcal{F} \to \mathcal{G}$$

such that the restriction to $\mathcal{F} = \mathrm{h}_{\mathrm{S}}(\mathrm{S}) \times \mathcal{F}$ along i_0 (resp. i_1) coincides with f (resp. g). We say that f and g are $\mathbf{A}^1$*-homotopic* if there exists a sequence of elementary $\mathbf{A}^1$-homotopies connecting them. Note that in this case the induced morphisms $\mathrm{L}_{\mathbf{A}^1}(\mathcal{F}) \rightrightarrows \mathrm{L}_{\mathbf{A}^1}(\mathcal{G})$ coincide.

A morphism $f : \mathcal{F} \to \mathcal{G}$ of spaces over S is called a *strict* $\mathbf{A}^1$*-homotopy equivalence* if there exists a morphism $g : \mathcal{G} \to \mathcal{F}$ such that the composites $f \circ g$ and $g \circ f$ are $\mathbf{A}^1$-homotopic to the identities. Note that strict $\mathbf{A}^1$-homotopy equivalences are $\mathbf{A}^1$-homotopy equivalences.

[5] In an absolute sense, i.e. affine over $\mathrm{Spec}(\mathbf{Z})$.

2.3.3. Let $\Delta^\bullet_\mathrm{S}$ denote the cosimplicial fibred space over S associated to the interval $\mathbf{A}^1_\mathrm{S}$, in the sense of [MV99, §2.3]. Hence $\Delta^n_\mathrm{S} \approx \mathbf{A}^n_\mathrm{S}$ for each n. Then we have:

Proposition 2.3.4. *(i) For each space $\mathcal{F}$ over* S *there is a canonical identification*

$$\mathrm{L}_{\mathbf{A}^1}(\mathcal{F}) = \varinjlim_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \underline{\mathrm{Hom}}_\mathrm{S}(\mathrm{h}_\mathrm{S}(\Delta^n_\mathrm{S}), \mathcal{F}) \tag{2.7}$$

and in particular, for each smooth S*-scheme* X*, there are canonical isomorphisms of spaces*

$$\Gamma(\mathrm{X}, \mathrm{L}_{\mathbf{A}^1}(\mathcal{F})) = \varinjlim_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \Gamma(\Delta^n_\mathrm{S} \underset{\mathrm{S}}{\times} \mathrm{X}, \mathcal{F}).$$

(ii) The category $\mathcal{S}pc_{\mathbf{A}^1}(\mathrm{S})$ has universality of colimits.

(iii) The $\mathbf{A}^1$-localization functor $\mathrm{L}_{\mathbf{A}^1}$ commutes with finite products.

Proof. The proof of (i) is just as in [MV99, §2.3]. For claim (ii), it suffices to show that the $\mathbf{A}^1$-localization functor $\mathrm{L}_{\mathbf{A}^1}$ is locally cartesian, i.e. the canonical morphism

$$\mathrm{L}_{\mathbf{A}^1}(\mathcal{F}' \underset{\mathcal{F}}{\times} \mathcal{G}) \to \mathcal{F}' \underset{\mathcal{F}}{\times} \mathrm{L}_{\mathbf{A}^1}(\mathcal{G})$$

is invertible for any morphism of $\mathbf{A}^1$-homotopy invariant spaces $\mathcal{F}' \to \mathcal{F}$ and any morphism $\mathcal{G} \to \mathcal{F}$. This follows immediately from (i). Claim (iii) follows from the fact that the colimit in (i) is sifted (as sifted colimits commute with finite products). □

Remark 2.3.5. From [Hoy17, Prop. 3.3] one deduces a version of the above proposition where the colimit in (i) is indexed on the sifted category of compositions of morphisms of the form $\mathbf{A}^1 \times \mathrm{X} \to \mathrm{X}$. This version is in fact sufficient for our purposes.

2.4. Motivic spaces.

2.4.1. We say that a space $\mathcal{F}$ over S is *motivic* if it is Nisnevich-local and $\mathbf{A}^1$-homotopy invariant.

2.4.2. We denote by $\mathcal{H}(\mathrm{S})$ the full subcategory of $\mathcal{S}pc(\mathrm{S})$ spanned by motivic spaces over S.

This is an accessible localization of $\mathcal{S}pc(\mathrm{S})$: the inclusion admits a left adjoint

$$\mathrm{L}_{\mathrm{mot}} : \mathcal{S}pc(\mathrm{S}) \to \mathcal{H}(\mathrm{S})$$

which can be described as the transfinite composite

$$\mathrm{L}_{\mathrm{mot}}(\mathcal{F}) = \varinjlim_{n\geqslant 0} (\mathrm{L}_{\mathbf{A}^1} \circ \mathrm{L}_{\mathrm{Nis}})^{\circ n}(\mathcal{F}). \tag{2.8}$$

This follows from the fact that the properties of Nisnevich-locality and $\mathbf{A}^1$-homotopy invariance are stable by filtered colimits.

In particular, $\mathcal{H}(\mathrm{S})$ is an arena.

2.4.3. We will say that a morphism $\mathcal{F}' \to \mathcal{F}$ of spaces over S is a *motivic equivalence* if the induced morphism

$$\mathrm{L}_{\mathrm{mot}}(\mathcal{F}') \to \mathrm{L}_{\mathrm{mot}}(\mathcal{F})$$

is invertible, or equivalently if for each motivic space $\mathcal{G}$, the induced morphism of spaces

$$\mathrm{Maps}(\mathcal{G}, \mathcal{F}') \to \mathrm{Maps}(\mathcal{G}, \mathcal{F})$$

is invertible.

We will say that a space $\mathcal{F}$ over S is *motivically contractible* if the canonical morphism $\mathcal{F} \to e_\mathrm{S}$ is a motivic equivalence, i.e. the motivic localization $\mathrm{L}_{\mathrm{mot}}(\mathcal{F})$ is a fibred space.

2.4.4. Let $\mathbf{C}$ be an arena. We say that a functor $\mathrm{H} : \mathcal{S}m_{/\mathrm{S}} \to \mathbf{C}$ satisfies Nisnevich excision if it sends any Nisnevich square, over a smooth S-scheme X, to a cocartesian square in $\mathbf{C}$. (One can also define the property of Nisnevich descent for H, in a similar way.)

We say that H satisfies $\mathbf{A}^1$-homotopy invariance if for each smooth S-scheme X, the canonical morphism

$$\mathrm{H}(\mathrm{X} \times \mathbf{A}^1) \to \mathrm{H}(\mathrm{X})$$

is invertible in $\mathbf{C}$.

Let $\mathcal{F}unct_{\mathrm{mot}}(\mathcal{S}m_{/\mathrm{S}}, \mathbf{C})$ denote the full subcategory of $\mathcal{F}unct(\mathcal{S}m_{/\mathrm{S}}, \mathbf{C})$ spanned by functors that satisfy Nisnevich excision (or equivalently, Nisnevich descent) and $\mathbf{A}^1$-homotopy invariance. We have the following universal property for the category $\mathcal{H}(\mathrm{S})$:

Theorem 2.4.5. *Let $\mathbf{C}$ be an arena. For any scheme* S*, the canonical functor*

$$\mathcal{F}unct_!(\mathcal{H}(\mathrm{S}), \mathbf{C}) \to \mathcal{F}unct_{\mathrm{mot}}(\mathcal{S}m_{/\mathrm{S}}, \mathbf{C}), \tag{2.9}$$

given by restriction along the functor $\mathcal{S}m_{/\mathrm{S}} \to \mathcal{H}(\mathrm{S})$, is an equivalence of categories.

Proof. Let $\mathcal{F}unct_{!,\mathrm{mot}}(\mathcal{S}pc(\mathrm{S}), \mathbf{C})$ denote the full subcategory of $\mathcal{F}unct_!(\mathcal{S}pc(\mathrm{S}), \mathbf{C})$ spanned by functors that send motivic equivalences to isomorphisms in $\mathbf{C}$. Under the equivalence (see [Chap. 0, Proposition 2.3.7])

$$\mathcal{F}unct_!(\mathcal{S}pc(\mathrm{S}), \mathbf{C}) \xrightarrow{\sim} \mathcal{F}unct(\mathcal{S}m_{/\mathrm{S}}, \mathbf{C}),$$

it is clear that the full subcategory $\mathcal{F}unct_{!,\mathrm{mot}}(\mathcal{S}pc(\mathrm{S}), \mathbf{C})$ identifies with $\mathcal{F}unct_{\mathrm{mot}}(\mathcal{S}m_{/\mathrm{S}}, \mathbf{C})$.

Hence by [Chap. 0, Proposition 2.6.17] we have canonical equivalences

$$\mathcal{F}unct_!(\mathcal{H}(\mathrm{S}), \mathbf{C}) \xrightarrow{\sim} \mathcal{F}unct_{!,\mathrm{mot}}(\mathcal{S}pc(\mathrm{S}), \mathbf{C}) \xrightarrow{\sim} \mathcal{F}unct_{\mathrm{mot}}(\mathcal{S}m_{/\mathrm{S}}, \mathbf{C})$$

as claimed. □

2.4.6. We have:

Lemma 2.4.7. *The localization functor $\mathrm{L}_{\mathrm{mot}}$ commutes with finite products.*

Proof. This follows from the formula (2.8): the functors $\mathrm{L}_{\mathrm{Nis}}$ and $\mathrm{L}_{\mathbf{A}^1}$ both commute with finite products, and filtered colimits commute with finite products in the topos $\mathcal{S}pc(\mathrm{S})$. □

By adjunction it follows that for a motivic space $\mathcal{F}$, the internal hom space $\underline{\mathrm{Hom}}_{\mathrm{S}}(\mathcal{G}, \mathcal{F})$ is motivic for any space $\mathcal{G}$. In particular:

Corollary 2.4.8. *The category $\mathcal{H}(\mathrm{S})$ is cartesian closed.*

In particular we get a cartesian monoidal structure on the arena $\mathcal{H}(\mathrm{S})$, which is the restriction of the symmetric monoidal structure on $\mathcal{S}pc(\mathrm{S})$.

2.4.9. Since the topos $\mathcal{S}pc_{\mathrm{Nis}}(\mathrm{S})$ has universality of colimits, as does $\mathcal{S}pc_{\mathbf{A}^1}(\mathrm{S})$ by [Chap. 1, Proposition 2.3.4], we have:

Proposition 2.4.10. *The category $\mathcal{H}(\mathrm{S})$ has universality of colimits.*

2.4.11. For each smooth S-scheme X, let $\mathrm{M}_{\mathrm{S}}(\mathrm{X})$ denote the motivic localization $\mathrm{L}_{\mathrm{mot}}(\mathrm{h}_{\mathrm{S}}(\mathrm{X}))$. [Chap. 1, Proposition 2.2.9] implies directly:

Proposition 2.4.12. *The category $\mathcal{H}(\mathrm{S})$ is generated under sifted colimits by the spaces $\mathrm{M}_{\mathrm{S}}(\mathrm{X})$, where* X *is a smooth* S*-scheme which is* affine *(over* $\mathrm{Spec}(\mathbf{Z})$*).*

2.4.13. Suppose the scheme S is classical, and noetherian of finite Krull dimension (in the classical sense). In this case there is an ordinary category of motivic spaces constructed by Morel–Voevodsky [MV99], which can be viewed as the underlying ordinary category of an $(\infty, 1)$-category (see [Rob14] or [Hoy15, Appendix C]).

We have:

Proposition 2.4.14. *If* S *is a classical noetherian scheme of finite Krull dimension, then the* $(\infty, 1)$*-category* $\mathcal{H}(\mathrm{S})$ *coincides with the Morel–Voevodsky* $(\infty, 1)$*-category of motivic spaces over* S*.*

Proof. This follows directly from the observation that the site $\mathcal{S}m_{/\mathrm{S}}$ is equivalent to the site of *classical* smooth S-schemes. Indeed, any smooth S-scheme X is flat over S, hence itself classical. □

3. Pointed motivic spaces

3.1. Pointed fibred spaces.

3.1.1. A *pointed (fibred) space* over a scheme S is a pointed object in the category $\mathcal{S}pc(\mathrm{S})$, i.e. a pair $(\mathcal{F}, x)$, with $\mathcal{F}$ a fibred space over S, and $x : e_{\mathrm{S}}^{\mathcal{S}pc} \to \mathcal{F}$ a morphism from the terminal object.

By definition it admits a zero object $\mathrm{pt}_{\mathrm{S}}^{\mathcal{S}pc_\bullet} := (e_{\mathrm{S}}^{\mathcal{S}pc}, \mathrm{id})$, where $\mathrm{id} : e_{\mathrm{S}}^{\mathcal{S}pc} \to e_{\mathrm{S}}^{\mathcal{S}pc}$ is the identity morphism.

We will write $\mathcal{S}pc(\mathrm{S})_\bullet$ for the arena of pointed spaces over S. By [Lur16, Ex. 4.8.1.20, Prop. 4.8.2.11], it has a canonical structure of $\mathcal{S}pc_\bullet$-module arena, and is canonically equivalent to the base change $\mathcal{S}pc(\mathrm{S}) \otimes_{\mathcal{S}pc} \mathcal{S}pc_\bullet$.

3.1.2. Consider the forgetful functor sending a pointed space $(\mathcal{F}, x)$ to its underlying space $\mathcal{F}$. This admits a left adjoint, which freely adjoins a point to $\mathcal{F}$; that is, it is given on objects by the assignment

$$\mathcal{F} \mapsto \mathcal{F}_+ := (\mathcal{F} \sqcup e_{\mathrm{S}}^{\mathcal{S}pc}, x)$$

where x is the canonical point.

For a smooth S-scheme X, we write $\mathrm{h}_{\mathrm{S}}^\bullet(\mathrm{X}) := \mathrm{h}_{\mathrm{S}}(\mathrm{X})_+$ for the pointed space represented by X.

3.1.3. It is clear that $\mathcal{S}pc(\mathrm{S})_\bullet$ is equivalent to the category of modules over the monad with underlying endofunctor $\mathcal{F} \mapsto \mathcal{F} \sqcup e_{\mathrm{S}}^{\mathcal{S}pc}$. Since the latter commutes with contractible colimits, it follows that the forgetful functor $(\mathcal{F}, x) \mapsto \mathcal{F}$ is conservative[6], and preserves and reflects contractible colimits.

This monadic description also implies that every pointed space can be written as a colimit of a simplicial diagram with each term in the essential image of $\mathcal{F} \mapsto \mathcal{F}_+$:

Lemma 3.1.4. *The category* $\mathcal{S}pc(\mathrm{S})_\bullet$ *is generated under sifted colimits by objects of the form* $\mathcal{F}_+$*, where* $\mathcal{F}$ *is a space over* S*.*

[6] Recall that a functor is *conservative* if it reflects isomorphisms.

3.1.5. The cartesian monoidal structure on the arena $\mathcal{S}pc(\mathrm{S})$ induces a monoidal structure on $\mathcal{S}pc(\mathrm{S})_\bullet$ (see [Rob15, Cor. 2.32]):

Lemma 3.1.6. *The arena $\mathcal{S}pc(\mathrm{S})_\bullet$ admits a canonical symmetric monoidal structure, which is uniquely characterized by the fact that the functor $\mathcal{S}pc(\mathrm{S}) \to \mathcal{S}pc(\mathrm{S})_\bullet$ is symmetric monoidal. Further, we have the following universal property:*

Given any symmetric monoidal morphism of arenas $u : \mathcal{S}pc(\mathrm{S}) \to \mathbf{C}$, with $\mathbf{C}$ pointed, there exists a unique symmetric monoidal morphism of arenas $\tilde{u} : \mathcal{S}pc(\mathrm{S})_\bullet \to \mathbf{C}$, and an isomorphism $\tilde{u} \circ (-)_+ \approx u$.

We will write $\otimes_{\mathrm{S}}^{\mathcal{S}pc_\bullet}$ for the monoidal product, and $\underline{\mathrm{Hom}}_{\mathrm{S}}^{\mathcal{S}pc_\bullet}$ for the internal hom. The monoidal unit is $\mathbf{1}_{\mathrm{S}}^{\mathcal{S}pc_\bullet} := (e_{\mathrm{S}}^{\mathcal{S}pc})_+ = \mathrm{h}_{\mathrm{S}}^\bullet(\mathrm{S})$.

3.1.7. Given a pointed space $\mathcal{T}$ over S, the $\mathcal{T}$-*suspension* endofunctor $\Sigma_{\mathcal{T}} := \Sigma_{\mathcal{T},\mathrm{S}} : \mathcal{S}pc(\mathrm{S})_\bullet \to \mathcal{S}pc(\mathrm{S})_\bullet$ is defined by the assignment

$$(\mathcal{F}, x) \mapsto (\mathcal{F}, x) \otimes_{\mathrm{S}} \mathcal{T}.$$

Dually, the $\mathcal{T}$-*loop space* endofunctor $\Omega_{\mathcal{T}} := \Omega_{\mathcal{T},\mathrm{S}}$ is given by

$$(\mathcal{F}, x) \mapsto \underline{\mathrm{Hom}}_{\mathrm{S}}(\mathcal{T}, (\mathcal{F}, x)).$$

These endofunctors form an adjunction $(\Sigma_{\mathcal{T}}, \Omega_{\mathcal{T}})$.

3.2. Pointed motivic spaces.

3.2.1. A pointed space $(\mathcal{F}, x)$ over S is Nisnevich-local, $\mathbf{A}^1$-homotopy invariant, or motivic, if the underlying space $\mathcal{F}$ has the respective property. We write $\mathcal{H}(\mathrm{S})_\bullet$ for the full subcategory of $\mathcal{S}pc(\mathrm{S})_\bullet$ spanned by motivic pointed spaces. Note that this is equivalent to the category of pointed objects in $\mathcal{H}(\mathrm{S})$.

We write $\mathrm{M}_{\mathrm{S}}^\bullet(\mathrm{X}) := \mathrm{M}_{\mathrm{S}}(\mathrm{X})_+$ for the pointed motivic space represented by a smooth S-scheme X.

3.2.2. The monadic description we have given of the category $\mathcal{S}pc(\mathrm{S})_\bullet$ also applies to $\mathcal{H}(\mathrm{S})_\bullet$, so that in particular we have:

Lemma 3.2.3. *The forgetful functor $(\mathcal{F}, x) \mapsto \mathcal{F}$, on the category of pointed motivic spaces over* S*, is conservative and preserves and reflects contractible colimits.*

3.2.4. The symmetric monoidal structure on the arena $\mathcal{S}pc(\mathrm{S})_\bullet$ restricts to one on $\mathcal{H}(\mathrm{S})_\bullet$, uniquely characterized by the fact that the morphism $\mathcal{H}(\mathrm{S}) \to \mathcal{H}(\mathrm{S})_\bullet$ is symmetric monoidal. As in [Chap. 1, Lemma 3.1.6] we have the following universal property:

Proposition 3.2.5. *Given any symmetric monoidal morphism of arenas $u : \mathcal{H}(\mathrm{S}) \to \mathbf{C}$, with $\mathbf{C}$ pointed, there exists a unique symmetric monoidal morphism $\tilde{u} : \mathcal{H}(\mathrm{S})_\bullet \to \mathbf{C}$, and an isomorphism $\tilde{u} \circ (-)_+ \approx u$.*

3.2.6. Note that the full subcategory $\mathcal{H}(\mathrm{S})_\bullet$ is a reflective localization of $\mathcal{S}pc(\mathrm{S})_\bullet$ at the small set of morphisms of the form

$$\mathrm{h}_{\mathrm{S}}^\bullet(\mathrm{X} \times \mathbf{A}^1) \to \mathrm{h}_{\mathrm{S}}^\bullet(\mathrm{X}) \tag{3.1}$$

for each smooth S-scheme X, and

$$\varinjlim_{n \in \mathbf{\Delta}^{\mathrm{op}}} \mathrm{h}_{\mathrm{S}}^\bullet(\check{\mathrm{C}}(\mathrm{X}_\alpha/\mathrm{X})) \to \mathrm{h}_{\mathrm{S}}^\bullet(\mathrm{X}) \tag{3.2}$$

for each smooth S-scheme X and Nisnevich covering family $(f_\alpha : \mathrm{X}_\alpha \to \mathrm{X})_\alpha$.

In particular we obtain localization functors $\mathrm{L}_{\mathrm{Nis}} := \mathrm{L}^{\mathcal{S}pc_\bullet}_{\mathrm{Nis,S}}$, $\mathrm{L}_{\mathbf{A}^1} := \mathrm{L}^{\mathcal{S}pc_\bullet}_{\mathbf{A}^1,\mathrm{S}}$, and

$$\mathrm{L}_{\mathrm{mot}} := \mathrm{L}^{\mathcal{S}pc_\bullet}_{\mathrm{mot,S}} : \mathcal{S}pc(\mathrm{S})_\bullet \to \mathcal{H}(\mathrm{S})_\bullet. \tag{3.3}$$

These admit descriptions which are completely analogous to their unpointed versions.

According to the universal property in [Chap. 1, Lemma 3.1.6], these localization functors are symmetric monoidal morphisms of arenas, characterized by commutativity with the functor $\mathcal{F} \mapsto \mathcal{F}_+$. In particular, we have

$$\mathrm{L}_{\mathrm{mot}}(\mathrm{h}^\bullet_{\mathrm{S}}(\mathrm{X})) = \mathrm{M}^\bullet_{\mathrm{S}}(\mathrm{X})$$

for each smooth S-scheme X.

3.2.7. As a result of [Chap. 1, Proposition 2.4.12] and [Chap. 1, Lemma 3.1.4], we have:

Proposition 3.2.8. *The category $\mathcal{H}(\mathrm{S})_\bullet$ is generated under sifted colimits by objects of the form $\mathrm{M}^\bullet_{\mathrm{S}}(\mathrm{X})$, for X a smooth S-scheme which is affine (over $\mathrm{Spec}(\mathbf{Z})$).*

4. Motivic spectra

4.1. Fibred spectra.

4.1.1. Let S be a scheme. Fix a pointed space $\mathcal{T}$ over S [7]. A *(fibred) $\mathcal{T}$-spectrum* over S is a $\mathcal{T}$-spectrum object in the category $\mathcal{S}pc(\mathrm{S})_\bullet$ of pointed spaces over $\mathcal{S}m_{/\mathrm{S}}$.

That is, a $\mathcal{T}$-spectrum is the data of a sequence $(\mathcal{F}_n)_{n\geqslant 0}$ of fibred pointed spaces over S and structural isomorphisms

$$\alpha_n : \mathcal{F}_n \xrightarrow{\sim} \Omega_{\mathcal{T}}(\mathcal{F}_{n+1})$$

for each integer $n \geqslant 0$, together with a homotopy coherent system of compatibilities between these isomorphisms.

4.1.2. We will write $\mathcal{S}pt_{\mathcal{T}}(\mathrm{S})$ for the arena of $\mathcal{T}$-spectra over S, which is by definition the limit of the cofiltered diagram

$$\cdots \xrightarrow{\Omega_{\mathcal{T}}} \mathcal{S}pc(\mathrm{S})_\bullet \xrightarrow{\Omega_{\mathcal{T}}} \mathcal{S}pc(\mathrm{S})_\bullet \tag{4.1}$$

in the category of $(\infty,1)$-categories and right adjoint functors.

Equivalently, this is the colimit of the filtered diagram

$$\mathcal{S}pc(\mathrm{S})_\bullet \xrightarrow{\Sigma_{\mathcal{T}}} \mathcal{S}pc(\mathrm{S})_\bullet \xrightarrow{\Sigma_{\mathcal{T}}} \cdots \tag{4.2}$$

in the category of arenas.

4.1.3. By construction, the adjunction $(\Sigma_{\mathcal{T}}, \Omega_{\mathcal{T}})$ at the level of pointed spaces gives rise to an *equivalence*

$$\Sigma^{\mathcal{S}pt}_{\mathcal{T}} : \mathcal{S}pt_{\mathcal{T}}(\mathrm{S}) \rightleftarrows \mathcal{S}pt_{\mathcal{T}}(\mathrm{S}) : \Omega^{\mathcal{S}pt}_{\mathcal{T}},$$

where $\Sigma^{\mathcal{S}pt}_{\mathcal{T}}$ is given by the assignment $(\mathcal{F}_n)_n \mapsto (\mathcal{F}_{n+1})_n$ (with the same structural isomorphisms), and $\Omega^{\mathcal{S}pt}_{\mathcal{T}}$ is given by $(\mathcal{F}_n)_n \mapsto (\Omega_{\mathcal{T}}(\mathcal{F}_{n+1}))_n$ (with the induced structural isomorphisms).

When there is no risk of ambiguity, we will write $\Sigma_{\mathcal{T}} := \Sigma^{\mathcal{S}pt}_{\mathcal{T}}$ and $\Omega_{\mathcal{T}} := \Omega^{\mathcal{S}pt}_{\mathcal{T}}$.

[7] For us $\mathcal{T}$ will be $\mathbf{S}^1$, $\mathbf{P}^1_{\mathrm{S}}$ or $\mathbf{S}^1 \otimes \mathbf{G}_m$.

4.1.4. Also by construction, we have for each $n \geqslant 0$ a canonical functor

$$\Omega^{\infty-n} : \mathcal{S}pt_{\mathcal{T}}(\mathrm{S}) \to \mathcal{S}pc(\mathrm{S})_{\bullet}$$

which is the projection to the nth term, with canonical isomorphisms $\Omega_{\mathcal{T}}\Omega_{\mathcal{T}}^{\infty-n-1} = \Omega_{\mathcal{T}}^{\infty-n}$.

Dually we have a canonical functor

$$\Sigma^{\infty-n} : \mathcal{S}pc(\mathrm{S})_{\bullet} \to \mathcal{S}pt_{\mathcal{T}}(\mathrm{S}),$$

left adjoint to $\Omega_{\mathcal{T}}^{\infty-n}$, with canonical isomorphisms $\Sigma_{\mathcal{T}}^{\infty-n-1}\Sigma_{\mathcal{T}}^{\infty} = \Sigma_{\mathcal{T}}^{\infty-n}$.

We will write $\Omega_{\mathcal{T}}^{\infty} := \Omega_{\mathcal{T}}^{\infty-0}$ and $\Sigma_{\mathcal{T}}^{\infty} := \Sigma_{\mathcal{T}}^{\infty-0}$. Note that there are canonical isomorphisms $\Sigma_{\mathcal{T}}^{\infty-n} = \Omega_{\mathcal{T}}^{n}\Sigma_{\mathcal{T}}^{\infty}$ and $\Omega_{\mathcal{T}}^{\infty-n} = \Omega_{\mathcal{T}}^{\infty}\Sigma_{\mathcal{T}}^{n}$ for each n.

4.1.5. Note that for $\mathcal{T} = \mathbf{S}^1$, the category of $\mathbf{S}^1$-spectra over S is equivalent to the category of presheves of spectra on the site $\mathcal{S}m_{/\mathrm{S}}$; we will identify the two implicitly. (Here $\mathbf{S}^1$ denotes the constant pointed space over S valued in the 1-sphere (with its canonical base point).)

4.1.6. For a smooth S-scheme X, we will write $\mathrm{h}_{\mathcal{T},\mathrm{S}}^{\infty}(\mathrm{X}) := \Sigma_{\mathcal{T},\mathrm{S}}^{\infty}(\mathrm{h}_{\mathrm{S}}^{\bullet}(\mathrm{X}))$ for the infinite suspension spectrum of the pointed motivic space represented by X. This is the $\mathcal{T}$-spectrum representing X, in the sense that for any $\mathcal{T}$-spectrum $\mathbb{E}$ there are canonical functorial isomorphisms of pointed spaces

$$\mathrm{Maps}_{\mathcal{S}pt_{\mathcal{T}}(\mathrm{S})}(\mathrm{h}_{\mathcal{T},\mathrm{S}}^{\infty}(\mathrm{X}), \mathbb{E}) \approx \Gamma(\mathrm{X}, \Omega_{\mathcal{T},\mathrm{S}}^{\infty}(\mathbb{E})). \tag{4.3}$$

4.1.7. By [Lur09b, Lem. 6.3.3.6], and the definition of $\mathcal{S}pt_{\mathcal{T}}(\mathrm{S})$ as the cofiltered limit of the diagram (4.1), we have:

Lemma 4.1.8. *The category $\mathcal{S}pt_{\mathcal{T}}(\mathrm{S})$ is generated under filtered colimits by objects of the form $\Sigma_{\mathcal{T}}^{\infty-n}(\mathcal{F})$, for $\mathcal{F}$ a pointed space over* S *and $n \geqslant 0$.*

4.1.9. Assume that for some $n \geqslant 2$, the pointed space $\mathcal{T}$ is *n-symmetric*, i.e. the cyclic permutation of $\mathcal{T}^{\otimes n}$ is homotopic to the identity morphism. In this case, the main result of [Rob15] endows the arena $\mathcal{S}pt_{\mathcal{T}}(\mathrm{S})$ with a canonical symmetric monoidal structure:

Lemma 4.1.10. *The arena $\mathcal{S}pt_{\mathcal{T}}(\mathrm{S})$ admits a canonical symmetric monoidal structure, and the functor $\Sigma_{\mathcal{T}}^{\infty}$ lifts to a symmetric monoidal morphism of arenas, which sends $\mathcal{T}$ to a monoidally invertible object of $\mathcal{S}pt_{\mathcal{T}}(\mathrm{S})$.*

Further, it satisfies the following universal property: given a symmetric monoidal arena $\mathbf{D}$, and a symmetric monoidal morphism $u : \mathcal{S}pc(\mathrm{S})_{\bullet} \to \mathbf{D}$ sending $\mathcal{T}$ to a monoidally invertible object in $\mathbf{D}$, there exists a unique symmetric monoidal morphism $\tilde{u} : \mathcal{S}pt_{\mathcal{T}}(\mathrm{S}) \to \mathbf{D}$ and an isomorphism $\tilde{u} \circ \Sigma_{\mathcal{T}}^{\infty} = u$.

We will write $\otimes_{\mathrm{S}} := \otimes_{\mathrm{S}}^{\mathcal{S}pt_{\mathcal{T}}}$ for the monoidal product in $\mathcal{S}pt_{\mathcal{T}}(\mathrm{S})$, and $\underline{\mathrm{Hom}}_{\mathrm{S}} := \underline{\mathrm{Hom}}_{\mathrm{S}}^{\mathcal{S}pt_{\mathcal{T}}}$ for the internal hom. The monoidal unit $\mathbf{1}_{\mathrm{S}} := \mathbf{1}_{\mathrm{S}}^{\mathcal{S}pt_{\mathcal{T}}}$ is the $\mathcal{T}$-spectrum $\mathrm{h}_{\mathcal{T},\mathrm{S}}^{\infty}(\mathrm{S})$.

4.1.11. We will abuse notation and write $\mathcal{T}$ also for the monoidally invertible object $\Sigma_{\mathcal{T}}^{\infty}(\mathcal{T})$. Let $\mathcal{T}^{\otimes 0} = \mathbf{1}_{\mathrm{S}}^{\mathcal{S}pt_{\mathcal{T}}}$ and let $\mathcal{T}^{\otimes(-1)}$ be a monoidal inverse to $\mathcal{T}$; for $n > 1$, write $\mathcal{T}^{\otimes(-n)} = (\mathcal{T}^{\otimes(-1)})^{\otimes n}$.

The universal property in [Chap. 1, Lemma 4.1.10] shows that there are canonical isomorphisms of functors $(\Sigma_{\mathcal{T}}^{\mathcal{S}pt})^{\circ n} = \mathcal{T}^{\otimes n} \otimes (-)$ and $(\Omega_{\mathcal{T}}^{\mathcal{S}pt})^{\circ n} = \mathcal{T}^{\otimes -n} \otimes (-)$ for each $n \geqslant 0$.

4.2. Motivic spectra.

4.2.1. We say that a $\mathcal{T}$-spectrum $\mathbb{E}$ over S satisfies Nisnevich descent or $\mathbf{A}^1$-homotopy invariance if for each $n \geqslant 0$, its nth component $\Omega_{\mathcal{T}}^{\infty-n}(\mathbb{E})$ satisfies the respective property (as a pointed fibred space).

A *motivic $\mathcal{T}$-spectrum* is a $\mathcal{T}$-spectrum satisfying Nisnevich descent and $\mathbf{A}^1$-homotopy invariance.

4.2.2. Let $\mathcal{SH}_{\mathcal{T}}(\mathrm{S})$ denote the category of motivic $\mathcal{T}$-spectra. This is equivalent to the category of $\mathrm{L}_{\mathrm{mot}}(\mathcal{T})$-spectra objects in the category of pointed motivic spaces over S. In particular it is an arena.

4.2.3. For a smooth S-scheme X, we will write $\mathrm{M}_{\mathrm{S}}^{\infty}(\mathrm{X}) := \mathrm{M}_{\mathcal{T},\mathrm{S}}^{\infty}(\mathrm{X})$ for the motivic $\mathcal{T}$-spectrum $\Sigma_{\mathcal{T}}^{\infty}(\mathrm{M}_{\mathrm{S}}^{\bullet}(\mathrm{X}))$, and similarly $\mathrm{M}_{\mathcal{T},\mathrm{S}}^{\infty-k}(\mathrm{X}) := \Sigma_{\mathcal{T}}^{\infty-k}(\mathrm{M}_{\mathrm{S}}^{\bullet}(\mathrm{X}))$ for $k \geqslant 0$.

By [Chap. 1, Proposition 2.4.12] and [Chap. 1, Lemma 4.1.8], we have:

Proposition 4.2.4. *The category $\mathcal{SH}_{\mathcal{T}}(\mathrm{S})$ is generated under sifted colimits by objects of the form $\Sigma_{\mathcal{T}}^{-n}(\mathrm{M}_{\mathcal{T},\mathrm{S}}^{\infty}(\mathrm{X}))$, for X a smooth S-scheme which is affine (over $\mathrm{Spec}(\mathbf{Z})$) and $n \geqslant 0$.*

4.2.5. Assuming that $\mathcal{T}$ is n-symmetric for some $n \geqslant 2$, the symmetric monoidal structure on the arena $\mathcal{S}pt_{\mathcal{T}}(\mathrm{S})$ ([Chap. 1, Lemma 4.1.10]) restricts to one on $\mathcal{SH}_{\mathcal{T}}(\mathrm{S})$.

4.2.6. Note that $\mathbb{E}$ satisfies Nisnevich descent (resp. $\mathbf{A}^1$-homotopy invariance) if and only if it is a local object with respect to the small set of morphisms

$$\varinjlim_{n\in\mathbf{\Delta}^{\mathrm{op}}} \Omega_{\mathcal{T}}^{k}\,\mathrm{h}_{\mathcal{T},\mathrm{S}}^{\infty}(\check{\mathrm{C}}(\mathrm{X}_\alpha/\mathrm{X})_n) \to \Omega_{\mathcal{T}}^{k}\,\mathrm{h}_{\mathcal{T},\mathrm{S}}^{\infty}(\mathrm{X}) \tag{4.4}$$

for every smooth S-scheme X and integer $k \geqslant 0$ (resp. the small set of morphisms

$$\Omega_{\mathcal{T}}^{k}\,\mathrm{h}_{\mathcal{T},\mathrm{S}}^{\infty}(\mathrm{X}\times\mathbf{A}^1) \to \Omega_{\mathcal{T}}^{k}\,\mathrm{h}_{\mathcal{T},\mathrm{S}}^{\infty}(\mathrm{X}) \tag{4.5}$$

for every smooth S-scheme X, Nisnevich covering family $(f_\alpha : \mathrm{X}_\alpha \to \mathrm{X})_\alpha$, and integer $k \geqslant 0$).

In other words, the full subcategory $\mathcal{SH}_{\mathcal{T}}(\mathrm{S})$ is an accessible localization of the arena $\mathcal{S}pt_{\mathcal{T}}(\mathrm{S})$, so that the inclusion admits a left adjoint

$$\mathrm{L}_{\mathrm{mot}} := \mathrm{L}_{\mathrm{mot},\mathrm{S}}^{\mathcal{S}pt_{\mathcal{T}}} : \mathcal{S}pt_{\mathcal{T}}(\mathrm{S}) \to \mathcal{SH}_{\mathcal{T}}(\mathrm{S}). \tag{4.6}$$

Similarly we have Nisnevich- and $\mathbf{A}^1$-localization functors denoted $\mathrm{L}_{\mathrm{Nis}}^{\mathcal{S}pt_{\mathcal{T}}}$ and $\mathrm{L}_{\mathbf{A}^1}^{\mathcal{S}pt_{\mathcal{T}}}$, respectively.

4.2.7. These localization functors admit descriptions completely analogous to their counterparts at the level of motivic spaces. For example,

$$\mathrm{L}_{\mathbf{A}^1}(\mathbb{E}) = \varinjlim_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \underline{\mathrm{Hom}}_{\mathrm{S}}^{\mathcal{S}pt_{\mathcal{T}}}(\mathrm{h}_{\mathcal{T},\mathrm{S}}^{\infty}(\Delta_{\mathrm{S}}^{n}), \mathbb{E}) \tag{4.7}$$

for every $\mathcal{T}$-spectrum $\mathbb{E}$.

Using the universal property ([Chap. 1, Lemma 4.1.10]), they define symmetric monoidal morphisms of arenas which can be characterized uniquely by commutativity with the functor $\Sigma_{\mathcal{T}}^{\infty}$. In particular, we have

$$\mathrm{L}_{\mathrm{mot}}(\mathrm{h}_{\mathcal{T},\mathrm{S}}^{\infty-k}(\mathrm{X})) = \mathrm{M}_{\mathcal{T},\mathrm{S}}^{\infty-k}(\mathrm{X})$$

for each $k \geqslant 0$.

4.2.8. As soon as the category $\mathcal{S}pt_{\mathcal{T}}(\mathrm{S})$ is *stable*[8], we have a canonical isomorphism

$$\mathrm{L}_{\mathrm{mot}}(\mathbb{E}) = \mathrm{L}_{\mathbf{A}^1}(\mathrm{L}_{\mathrm{Nis}}(\mathbb{E})) \tag{4.8}$$

for any $\mathcal{T}$-spectrum $\mathbb{E}$.

This follows from the fact that the condition of Nisnevich descent is defined by finite limits, and finite limits commute with colimits in stable $(\infty, 1)$-categories.

[8] This will be true in all the cases we consider, e.g. $\mathcal{T} = \mathbf{S}^1$, $\mathcal{T} = \mathbf{S}^1 \otimes (\mathrm{M}_{\mathrm{S}}(\mathbf{A}^{1,\times}_{\mathrm{S}}), 1) = (\mathrm{M}_{\mathrm{S}}(\mathbf{P}^1_{\mathrm{S}}), \infty)$.

5. Inverse and direct image functoriality

5.1. For motivic spaces.

5.1.1. Let $f : \mathrm{T} \to \mathrm{S}$ be a morphism of schemes. The direct image functor

$$f_*^{\mathcal{S}pc} : \mathcal{S}pc(\mathrm{T}) \to \mathcal{S}pc(\mathrm{S}) \tag{5.1}$$

is defined as restriction along the base change functor $\mathcal{S}m_{/\mathrm{S}} \to \mathcal{S}m_{/\mathrm{T}}$.

Its left adjoint $f^*_{\mathcal{S}pc}$, the inverse image functor, is given by left Kan extension (see [Chap. 0, Proposition 2.3.7]). Hence it is uniquely characterized by commutativity with small colimits and the formula

$$f^*_{\mathcal{S}pc}(\mathrm{h}_\mathrm{S}(\mathrm{X})) = \mathrm{h}_\mathrm{T}(\mathrm{X} \underset{\mathrm{S}}{\times} \mathrm{T}) \tag{5.2}$$

for smooth S-schemes X.

5.1.2. Note that the base change functor $\mathcal{S}m_{/\mathrm{S}} \to \mathcal{S}m_{/\mathrm{T}}$ preserves Nisnevich covering families and $\mathbf{A}^1$-projections. It follows that the inverse image functor $f^*_{\mathcal{S}pc}$ preserves Nisnevich-local equivalences and $\mathbf{A}^1$-homotopy equivalences.

By adjunction, its right adjoint $f_*^{\mathcal{S}pc}$ preserves Nisnevich-local and $\mathbf{A}^1$-homotopy invariant spaces and induces a functor $f_*^{\mathcal{H}} : \mathcal{H}(\mathrm{T}) \to \mathcal{H}(\mathrm{S})$. This admits a left adjoint $f_*^{\mathcal{H}}$ given by the formula

$$f^*_{\mathcal{H}}(\mathcal{F}) = \mathrm{L}_{\mathrm{mot}}(f^*_{\mathcal{S}pc}(\mathcal{F})).$$

5.1.3. Both the direct and inverse image functors are symmetric monoidal:

Lemma 5.1.4. *The functor $f_*^{\mathcal{S}pc}$ (resp. $f_*^{\mathcal{H}}$) admits a canonical symmetric monoidal structure.*

Proof. Since the respective symmetric monoidal structures are cartesian, it suffices to show that f_* commutes with finite products. In fact, it commutes with arbitrary small limits since it is a right adjoint. □

Lemma 5.1.5. *The functor $f^*_{\mathcal{S}pc}$ (resp. $f^*_{\mathcal{H}}$) admits a canonical symmetric monoidal structure.*

Proof. By adjunction from [Chap. 1, Lemma 5.1.4], we obtain a canonical structure of colax symmetric monoidal functor on f^*. That is, there are canonical morphisms

$$f^*(\mathcal{F} \underset{\mathrm{S}}{\times} \mathcal{G}) \to f^*(\mathcal{F}) \underset{\mathrm{T}}{\times} f^*(\mathcal{G}) \tag{5.3}$$

for any two spaces $\mathcal{F}$ and $\mathcal{G}$ over S. It suffices to show that these morphisms are invertible.

Since $f^*_{\mathcal{S}pc}$ commutes with small colimits, and the cartesian product commutes with small colimits in each argument, one reduces to the case of representables, in which case the claim is clear. For $f^*_{\mathcal{H}}$, the claim follows from the first because motivic localization commutes with finite products ([Chap. 1, Proposition 2.4.7]). □

5.2. For pointed motivic spaces.

5.2.1. Let $f : \mathrm{T} \to \mathrm{S}$ be a morphism of schemes. Since $f_*^{\mathcal{S}pc}$ preserves the terminal object, it induces a functor $f_*^{\mathcal{S}pc_\bullet} : \mathcal{S}pc(\mathrm{T}) \to \mathcal{S}pc(\mathrm{S})$ given on objects by the formula

$$f_*^{\mathcal{S}pc_\bullet}(\mathcal{G}, y) = (f_*^{\mathcal{S}pc}(\mathcal{G}), f_*^{\mathcal{S}pc}(y)).$$

Its left adjoint $f^*_{\mathcal{S}pc_\bullet} : \mathcal{S}pc(\mathrm{T}) \to \mathcal{S}pc(\mathrm{S})$ is uniquely characterized, according to [Chap. 1, Lemma 3.1.4], by the fact that it commutes with sifted colimits and with the functor $\mathcal{F} \mapsto \mathcal{F}_+$:

$$f^*_{\mathcal{S}pc_\bullet}(\mathcal{F}_+) = f^*_{\mathcal{S}pc}(\mathcal{F})_+ \tag{5.4}$$

for any space $\mathcal{F}$ over S.

Explicitly, it is given on objects by the formula

$$f^*_{\mathcal{S}pc_\bullet}(\mathcal{F}, x) = (f^*_{\mathcal{S}pc}(\mathcal{F}), f^*_{\mathcal{S}pc}(x))$$

for each pointed space $(\mathcal{F}, x)$ over S.

Lemma 5.2.2. *The inverse image functor $f^*_{\mathcal{S}pc_\bullet}$ admits a canonical symmetric monoidal structure.*

Proof. This follows directly from the universal property [Chap. 1, Lemma 3.1.6] and the formula (5.4). □

5.2.3. The direct image functor $f_*^{\mathcal{S}pc_\bullet}$ preserves the properties of Nisnevich descent and $\mathbf{A}^1$-homotopy invariance, and induces a functor $f_*^{\mathcal{H}}$. Its left adjoint $f^*_{\mathcal{H}_\bullet}$ is given by composing $f^*_{\mathcal{S}pc_\bullet}$ with the motivic localization functor:

$$f^*_{\mathcal{H}_\bullet} := \mathrm{L}_{\mathrm{mot}} f^*_{\mathcal{S}pc_\bullet}. \tag{5.5}$$

5.3. For motivic spectra. In this paragraph we fix, for each scheme S, a pointed space $\mathcal{T}_\mathrm{S}$ over S, together with isomorphisms $f^*_{\mathcal{S}pc_\bullet}(\mathcal{T}_\mathrm{S}) \to \mathcal{T}_\mathrm{T}$ for any morphism $f : \mathrm{T} \to \mathrm{S}$, and a homotopy coherent system of compatibilities between these isomorphisms.

Such a datum defines an object of the limit $\varprojlim_\mathrm{S} \mathcal{S}pc(\mathrm{S})_\bullet$ over all schemes S. Of course, this limit is equivalent to the category $\mathcal{S}pc(\mathrm{Spec}(\mathbf{Z}))_\bullet$, so that choosing such a datum is equivalent to choosing a pointed space over $\mathrm{Spec}(\mathbf{Z})$ and defining $\mathcal{T}_\mathrm{S}$ to be its inverse image over each scheme S.

For simplicity, we will drop $\mathcal{T}_\mathrm{S}$ from the subscripts in the notation $\mathcal{S}pt(\mathrm{S})$, $\mathcal{SH}(\mathrm{S})$, Σ^∞, Ω^∞, etc., when there is no risk of confusion.

5.3.1. Let $f : \mathrm{T} \to \mathrm{S}$ be a morphism of schemes. Since $f^*_{\mathcal{S}pc_\bullet}$ is monoidal ([Chap. 1, Lemma 5.2.2]), it commutes with $\mathcal{T}$-suspensions.

We let $f^*_{\mathcal{S}pt}$ denote the unique morphism of arenas making the diagram

$$\begin{tikzcd}
\mathcal{S}pc(\mathrm{S})_\bullet \arrow[r, "\Sigma_\mathcal{T}"] \arrow[d, "f^*_{\mathcal{S}pc_\bullet}"] & \mathcal{S}pc(\mathrm{S})_\bullet \arrow[r, "\Sigma_\mathcal{T}"] \arrow[d, "f^*_{\mathcal{S}pc_\bullet}"] & \cdots \arrow[r] & \mathcal{S}pt(\mathrm{S}) \arrow[d, dashed, "f^*_{\mathcal{S}pt}"] \\
\mathcal{S}pc(\mathrm{T})_\bullet \arrow[r, "\Sigma_\mathcal{T}"] & \mathcal{S}pc(\mathrm{T})_\bullet \arrow[r, "\Sigma_\mathcal{T}"] & \cdots \arrow[r] & \mathcal{S}pt(\mathrm{T})
\end{tikzcd}$$

commute. That is, it is the unique morphism of arenas which commutes with $\Sigma^\infty_\mathcal{T}$.

Using the universal property stated in [Chap. 1, Lemma 4.1.10], we get:

Lemma 5.3.2. *The functor $f^*_{\mathcal{S}pt}$ admits a canonical symmetric monoidal structure.*

5.3.3. Alternatively, we can use [Chap. 1, Lemma 4.1.8] to describe $f^*_{\mathcal{S}pt}$ as the unique functor which commutes with filtered colimits and with the functor $\Sigma^{-n}_{\mathcal{T}}\Sigma^{\infty}_{\mathcal{T}}$ for each $n \geqslant 0$:

$$f^*_{\mathcal{S}pt}(\Sigma^{-n}_{\mathcal{T}}\Sigma^{\infty}_{\mathcal{T}}(\mathcal{F})) = \Sigma^{-n}_{\mathcal{T}}\Sigma^{\infty}_{\mathcal{T}}(f^*_{\mathcal{S}pt}(\mathcal{F})) \tag{5.6}$$

for each pointed space $\mathcal{F}$ over S.

5.3.4. Let $f^{\mathcal{S}pt}_*$ be the right adjoint of $f^*_{\mathcal{S}pt}$. This can be described as the unique limit-preserving functor which makes the diagram

$$\begin{tikzcd}
\mathcal{S}pt(\mathrm{T}) \arrow[r] \arrow[d, dashed, "f^{\mathcal{S}pt}_*"] & \cdots \arrow[r, "\Omega_{\mathcal{T}}"] & \mathcal{S}pc(\mathrm{T})_{\bullet} \arrow[r, "\Omega_{\mathcal{T}}"] \arrow[d, "f^{\mathcal{S}pc_{\bullet}}_*"] & \mathcal{S}pc(\mathrm{T})_{\bullet} \arrow[d, "f^{\mathcal{S}pc_{\bullet}}_*"] \\
\mathcal{S}pt(\mathrm{S}) \arrow[r] & \cdots \arrow[r, "\Omega_{\mathcal{T}}"] & \mathcal{S}pc(\mathrm{S})_{\bullet} \arrow[r, "\Omega_{\mathcal{T}}"] & \mathcal{S}pc(\mathrm{S})_{\bullet}
\end{tikzcd}$$

commute, i.e. which commutes with Ω^{∞}:

$$\Omega^{\infty} f^{\mathcal{S}pt}_* = f^{\mathcal{S}pc_{\bullet}}_* \Omega^{\infty} \tag{5.7}$$

and is given on objects by the assignment

$$\mathbb{E} = (\mathcal{F}_n)_n \mapsto f_*(\mathbb{E}) = (f^{\mathcal{S}pc_{\bullet}}_*(\mathcal{F}_n))_n.$$

5.3.5. The direct image functor $f^{\mathcal{S}pt}_*$ preserves motivic spectra and induces a functor $f^{\mathcal{SH}}_*$. We let $f^*_{\mathcal{SH}}$ be its left adjoint, the symmetric monoidal functor $\mathrm{L}_{\mathrm{mot}} f^*_{\mathcal{S}pt}$. This is the unique symmetric monoidal morphism of arenas which commutes with $\Sigma^{\infty}_{\mathcal{T}} : \mathcal{H}(\mathrm{S})_{\bullet} \to \mathcal{SH}(\mathrm{S})$.

6. Smooth morphisms

6.1. The functor $p_{\sharp}$.

6.1.1. Let $p : \mathrm{X} \to \mathrm{S}$ be a smooth morphism. In this case the base change functor admits a right adjoint, the forgetful functor $\mathcal{S}m_{/\mathrm{X}} \to \mathcal{S}m_{/\mathrm{S}}$:

$$(\mathrm{Y} \to \mathrm{X}) \mapsto (\mathrm{Y} \to \mathrm{X} \xrightarrow{p} \mathrm{S}).$$

It follows that the functor $p^*_{\mathcal{S}pc}$ coincides with restriction along the forgetful functor, and admits a left adjoint

$$p^{\mathcal{S}pc}_{\sharp} : \mathcal{S}pc(\mathrm{T}) \to \mathcal{S}pc(\mathrm{S})$$

which is defined by left Kan extension (see [Chap. 0, Proposition 2.3.7]), and hence is uniquely characterized by commutativity with small colimits and the formula

$$p^{\mathcal{S}pc}_{\sharp}(\mathrm{h}_{\mathrm{X}}(\mathrm{Y})) = \mathrm{h}_{\mathrm{S}}(\mathrm{Y}). \tag{6.1}$$

for smooth X-schemes Y.

6.1.2. Since the forgetful functor $\mathcal{S}m_{/\mathrm{X}} \to \mathcal{S}m_{/\mathrm{S}}$ preserves Nisnevich covering families and $\mathbf{A}^1$-projections, it follows that $p^{\mathcal{S}pc}_{\sharp}$ preserves Nisnevich-local equivalences and $\mathbf{A}^1$-homotopy equivalences.

In particular its right adjoint $p^*_{\mathcal{S}pc}$ preserves Nisnevich descent and $\mathbf{A}^1$-homotopy invariance, and induces a morphism $p^*_{\mathcal{H}}$ on motivic spaces.

Its left adjoint $p^{\mathcal{H}}_{\sharp}$ is given by applying $p^{\mathcal{S}pc}_{\sharp}$ and then the localization functor $\mathrm{L}_{\mathrm{mot}}$:

$$p^{\mathcal{H}}_{\sharp}(\mathcal{F}) = \mathrm{L}_{\mathrm{mot}}(p^{\mathcal{S}pc}_{\sharp}(\mathcal{F})).$$

6.1.3. Let $p : \mathrm{X} \hookrightarrow \mathrm{S}$ be a smooth morphism. By [Chap. 1, Lemma 5.1.5], the functors $p^*_{\mathcal{S}pc}$ and $p^*_{\mathcal{H}}$ admit canonical symmetric monoidal structures, so that their respective left adjoints $p_\sharp^{\mathcal{S}pc}$ and $p_\sharp^{\mathcal{H}}$ admit colax symmetric monoidal structures.

If p is an *open immersion*, then these monoidal structures are strict:

Proposition 6.1.4. *Let $j : \mathrm{U} \hookrightarrow \mathrm{X}$ be a quasi-compact open immersion. Then the canonical colax symmetric monoidal structure on the functor $j_\sharp^{\mathcal{S}pc}$ (resp. $j_\sharp^{\mathcal{H}}$) is strict.*

Proof. It suffices to show that the canonical morphisms

$$j_\sharp^{\mathcal{S}pc}(\mathcal{F} \underset{\mathrm{U}}{\times} \mathcal{G}) \to j_\sharp^{\mathcal{S}pc}(\mathcal{F}) \underset{\mathrm{S}}{\times} j_\sharp^{\mathcal{S}pc}(\mathcal{G})$$

are invertible for all spaces $\mathcal{F}$ and $\mathcal{G}$ on U. Since $j_\sharp^{\mathcal{S}pc}$ commutes with small colimits, and the cartesian product commutes with small colimits in each argument, one reduces to the case of representables.

Then the claim follows from the fact that the fibred products $\mathrm{X} \times_{\mathrm{U}} \mathrm{Y}$ and $\mathrm{X} \times_{\mathrm{S}} \mathrm{Y}$ are canonically identified (since j is a monomorphism, i.e. its diagonal morphism is invertible). As above, the claim for $j_\sharp^{\mathcal{H}}$ follows the fact that motivic localization commutes with finite products. □

6.1.5. Let $(f_\alpha : \mathrm{S}_\alpha \to \mathrm{S})_\alpha$ be a Nisnevich covering family. Given a morphism of motivic spaces over S, the following proposition says that it is invertible if and only if its inverse image on each S_α is invertible:

Proposition 6.1.6 (Nisnevich separation)**.** *Let* S *be a scheme. For any Nisnevich covering family $(p_\alpha : \mathrm{S}_\alpha \to \mathrm{S})_\alpha$, the family of inverse image functors $(p_\alpha)^*_{\mathcal{H}} : \mathcal{H}(\mathrm{S}) \to \mathcal{H}(\mathrm{S}_\alpha)$ is conservative.*

This is in fact true at the level of Nisnevich-local spaces, which is what we will prove.

Proof. Let $\varphi : \mathcal{F}_1 \to \mathcal{F}_2$ be a morphism of Nisnevich-local spaces on S, and suppose that the following condition holds:

($*$) For each α, the morphism $(p_\alpha)^*_{\mathrm{Nis}}(\mathcal{F}_1) \to (p_\alpha)^*_{\mathrm{Nis}}(\mathcal{F}_2)$ is invertible.

The claim is that under this assumption, the morphism

$$\Gamma(\mathrm{X}, \mathcal{F}_1) \to \Gamma(\mathrm{X}, \mathcal{F}_2)$$

is invertible for every smooth S-scheme X.

Since $\mathcal{F}_i$ satisfy Nisnevich descent, it suffices to show that the morphism

$$\Gamma(\mathrm{X}_\alpha, \mathcal{F}_1) \to \Gamma(\mathrm{X}_\alpha, \mathcal{F}_2) \tag{6.2}$$

is invertible for each α, where X_α is the base change of X along p_α.

Since $\mathrm{h}_{\mathrm{S}}(\mathrm{X}_\alpha) = (p_\alpha)_\sharp (p_\alpha)^*(\mathrm{h}_{\mathrm{S}}(\mathrm{X}))$, we have by adjunction

$$\Gamma(\mathrm{X}_\alpha, \mathcal{F}_i) = \Gamma(\mathrm{X}, (p_\alpha)_\sharp (p_\alpha)^* \mathcal{F}_i)$$

for each α and i.

Hence the claim follows from the assumption ($*$). □

6.2. Smooth base change formulas.

6.2.1. Suppose we have a cartesian square

$$\begin{tikzcd} \mathrm{T}' \arrow[r, "f'"] \arrow[d, "p'"] & \mathrm{S}' \arrow[d, "p"] \\ \mathrm{T} \arrow[r, "f"] & \mathrm{S} \end{tikzcd}$$

of schemes.

At the level of (motivic) spaces, pointed spaces, and spectra, there are canonical 2-morphisms

$$(p')_\sharp (f')^* \to f^* p_\sharp, \tag{6.3}$$

$$f_* p^* \to (p')^* (f')_*, \tag{6.4}$$

constructed in [Chap. 2, Paragraph 2.2].

The following says that $\mathcal{S}pc$ and $\mathcal{H}$ satisfy the *left base change property* along smooth morphisms (see *loc. cit.*):

Proposition 6.2.2. *If p and p' are smooth, then the 2-morphisms (6.3) and (6.4) are invertible at the level of spaces and motivic spaces.*

Proof. It suffices to consider the first 2-morphism; the second is its right transpose.

For fibred spaces, we note that the functors in question commute with small colimits, so that we may reduce to representable spaces, in which case the claim is obvious.

Similarly, for motivic spaces we use [Chap. 1, Proposition 2.4.12] to reduce to the case of motivic localizations of representable spaces. $\square$

6.2.3. Next we consider the case of pointed spaces. Then we have:

Proposition 6.2.4. *If p and p' are smooth, then the 2-morphisms (6.3) and (6.4) are invertible at the level of pointed spaces and pointed motivic spaces.*

Proof. By transposition it suffices to show that the canonical morphism $(p')_\sharp(f')^* \to f^*p_\sharp$ is invertible. Since the functors in question commute with colimits and with the functor $\mathcal{F} \mapsto \mathcal{F}_+$, the claim follows from [Chap. 1, Lemma 3.1.4] (resp. [Chap. 1, Proposition 3.2.8]) and smooth base change for unpointed spaces ([Chap. 1, Proposition 6.2.2]). $\square$

6.2.5. Fix a family of pointed fibred spaces $(\mathcal{T}_\mathrm{S})_\mathrm{S}$ as in [Chap. 1, Paragraph 5.3].

We have:

Proposition 6.2.6. *If p and p' are smooth, then the 2-morphisms (6.3) and (6.4) are invertible at the level of $\mathcal{T}$-spectra and motivic $\mathcal{T}$-spectra.*

Proof. This follows from [Chap. 1, Lemma 4.1.8] (resp. [Chap. 1, Proposition 4.2.4]) and smooth base change for pointed spaces ([Chap. 1, Proposition 6.2.4]). $\square$

6.3. Smooth projection formulas. Let $p : \mathrm{X} \to \mathrm{S}$ be a smooth morphism. Note that the symmetric monoidal functor $p^*_{\mathcal{S}pc}$ endows $\mathcal{S}pc(\mathrm{X})$ with a structure of $\mathcal{S}pc(\mathrm{S})$-module category.

The following verifies the *left projection formula* along smooth morphisms, in the sense of [Chap. 2, Paragraph 2.2]:

Proposition 6.3.1. *The functor $p_\sharp^{\mathcal{S}pc}$ (resp. $p_\sharp^{\mathcal{H}}$) lifts to a morphism of $\mathcal{S}pc(\mathrm{S})$-module categories (resp. $\mathcal{H}(\mathrm{S})$-module categories). In other words, there are canonical isomorphisms*

$$p_\sharp(\mathcal{G} \underset{\mathrm{X}}{\times} p^*(\mathcal{F})) \to p_\sharp(\mathcal{G}) \underset{\mathrm{S}}{\times} \mathcal{F} \tag{6.5}$$

and dually

$$\underline{\mathrm{Hom}}_{\mathrm{S}}(p_\sharp(\mathcal{G}), \mathcal{F}) \to p_\sharp \underline{\mathrm{Hom}}_{\mathrm{X}}(\mathcal{G}, p^*(\mathcal{F})) \tag{6.6}$$

for any fibred spaces (resp. motivic spaces) $\mathcal{F}$ *over* S *and* $\mathcal{G}$ *over* X.

We recall how to use the monoidal structure on p^* to construct the morphism (6.5):

The counit of the adjunction $(p_\sharp^{\mathcal{S}pc}, p^*_{\mathcal{S}pc})$ induces a canonical morphism

$$\mathcal{G} \underset{\mathrm{X}}{\times} p^*(\mathcal{F}) \to p^* p_\sharp(\mathcal{G}) \underset{\mathrm{X}}{\times} p^*(\mathcal{F}) \xrightarrow{\sim} p^*(p_\sharp(\mathcal{G}) \underset{\mathrm{S}}{\times} \mathcal{F})$$

which corresponds by adjunction to the morphism desired.

Proof. It suffices to show that the canonical morphism (6.5) is invertible. For fibred spaces, we may reduce to the case where the spaces $\mathcal{F}$ and $\mathcal{G}$ are representable, since the functions involved commute with small colimits. In this case the claim is clear. The case of motivic spaces is similar, using [Chap. 1, Proposition 2.4.12] to reduce to the case of motivic localizations of representable spaces. □

6.3.2. The following slightly more general formula, proved in exactly the same way, will also be useful:

Lemma 6.3.3. *Let* $p : \mathrm{X} \to \mathrm{S}$ *be a smooth morphism. Let* $\mathcal{G}$ *be a fibred space (resp. motivic space) over* X, *and* $\mathcal{F} \to \mathcal{F}'$ *a morphism of fibred spaces (resp. motivic spaces) over* S. *Then there is a canonical isomorphism*

$$p_\sharp(\mathcal{G} \underset{p^*(\mathcal{F}')}{\times} p^*(\mathcal{F})) \xrightarrow{\sim} p_\sharp(\mathcal{G}) \underset{\mathcal{F}'}{\times} \mathcal{F} \tag{6.7}$$

of fibred spaces (resp. motivic spaces) over S.

6.3.4. Similarly we get smooth projection formulas for pointed spaces and spectra. As above, the following statements are equivalent to formulas of the form (6.5) and (6.6).

Proposition 6.3.5. *The functor* $p_\sharp^{\mathcal{S}pc_\bullet}$ *(resp.* $p_\sharp^{\mathcal{H}_\bullet}$*) lifts to a morphism of* $\mathcal{S}pc(\mathrm{S})_\bullet$*-module categories (resp.* $\mathcal{H}(\mathrm{S})_\bullet$*-module categories).*

Proof. This follows from [Chap. 1, Lemma 3.1.4] (resp. [Chap. 1, Proposition 3.2.8]) and the smooth projection formula for unpointed spaces ([Chap. 1, Proposition 6.3.1]). □

Fix a family of pointed fibred spaces $(\mathcal{T}_{\mathrm{S}})_{\mathrm{S}}$ as in [Chap. 1, Paragraph 5.3]. Then we have:

Proposition 6.3.6. *The functor* $p_\sharp^{\mathcal{S}pt}$ *(resp.* $p_\sharp^{\mathcal{SH}}$*) lifts to a morphism of* $\mathcal{S}pt(\mathrm{S})$*-module categories (resp.* $\mathcal{SH}(\mathrm{S})$*-module categories).*

Proof. This follows from [Chap. 1, Lemma 4.1.8] (resp. [Chap. 1, Proposition 4.2.4]) and the smooth projection formula for pointed spaces ([Chap. 1, Proposition 6.3.5]). □

7. Closed immersions

7.1. A topological digression, I: local cocontinuity. In order to prove [Chap. 1, Proposition 7.3.2], the main result of this section, we will begin by making a small topological digression.

Recall that a functor u between sites is *topologically cocontinuous*[9] if the restriction of presheaves functor u^* preserves local equivalences. In this paragraph we introduce a slightly

[9]The term *cocontinuous* is used in [SGA 4].

weaker version of this condition, where the functor u is only "locally cocontinuous" with respect to a weaker topology. We will show that for such functors, the restriction functor u^* preserves local equivalences between sheaves for the weaker topology.

7.1.1. Let $\mathbf{C}$ be a small $(\infty,1)$-category. As in [Chap. 0, Sect. 2], we will write $\mathcal{P}(\mathbf{C})$ for the $(\infty,1)$-category of presheaves on $\mathbf{C}$, and $h : \mathbf{C} \hookrightarrow \mathcal{P}(\mathbf{C})$ for the Yoneda embedding. Given a topology τ on $\mathbf{C}$, we will write $\mathcal{S}h_\tau(\mathbf{C})$ for the subcategory of τ-sheaves, i.e. the presheaves $\mathcal{F}$ for which the canonical morphism

$$\mathcal{F}(c) \to \mathrm{Maps}(\mathrm{R}, \mathcal{F})$$

is invertible for all τ-covering sieves $\mathrm{R} \hookrightarrow h(c)$ of all objects $c \in \mathbf{C}$. We will write $\mathrm{inc}_\tau : \mathcal{S}h_\tau(\mathbf{C}) \hookrightarrow \mathcal{P}(\mathbf{C})$ for the inclusion, and $\mathrm{L}_\tau : \mathcal{P}(\mathbf{C}) \to \mathcal{S}h_\tau(\mathbf{C})$ for the left-exact left adjoint (the τ-localization functor).

Given a functor $u : \mathbf{C} \to \mathbf{D}$, we will write $u^* : \mathcal{P}(\mathbf{D}) \to \mathcal{P}(\mathbf{C})$ for the restriction of presheaves functor, and $u_!$ (resp. u_*) for the left adjoint (resp. right adjoint).

7.1.2. Let $(\mathbf{C}, \tau)$ and $(\mathbf{D}, \tau')$ be sites. Recall that a functor $u : \mathbf{C} \to \mathbf{D}$ is *topologically cocontinuous* if the following condition is satisfied:

(COC) For every τ'-covering sieve $\mathrm{R}' \hookrightarrow h(u(c))$, the sieve $\mathrm{R} \hookrightarrow h(c)$, generated by morphisms $c' \to c$ such that $h(u(c')) \to h(u(c))$ factors through R', is τ-covering.

Note that R can be described as the sieve

$$\mathrm{R} = u^*(\mathrm{R}') \underset{u^*(h(u(c)))}{\times} h(c) \hookrightarrow h(c) \tag{7.1}$$

obtained from $\mathrm{R}' \hookrightarrow h(u(c))$ by applying u^* and taking the base change along the unit morphism $h(c) \to u^* u_!(h(c)) = u^* h(u(c))$.

7.1.3. Let τ'_0 be a topology on $\mathbf{D}$ which is weaker than τ'. For simplicity we will assume that τ'_0 is subcanonical, so that representable presheaves are τ'_0-sheaves. Let $\mathrm{L}_{\tau'_0}$ denote the associated τ'_0-sheaf functor, left adjoint to the inclusion.

We will say that u is *τ'_0-locally topologically cocontinuous* if it satisfies the following weaker version of the condition (COC):

(COC') For every τ'-covering sieve $\mathrm{R}' \hookrightarrow h(u(c))$, the sieve $\mathrm{R} \hookrightarrow h(c)$, generated by morphisms $c' \to c$ such that $h(u(c')) \to h(u(c))$ factors through the *τ'_0-sheaf associated* to R', is τ-covering.

Note that R can be described as the sieve

$$\mathrm{R} := u^*(\mathrm{L}_{\tau'_0}(\mathrm{R}')) \underset{u^*(h(u(c)))}{\times} h(c) \hookrightarrow h(c) \tag{7.2}$$

obtained in the same way as (7.1) starting from $\mathrm{L}_{\tau'_0}(\mathrm{R}') \hookrightarrow h(u(c))$.

7.1.4. The following lemma is proved in exactly the same way as the analoguous result for topologically cocontinuous functors [SGA 4, Exp. III, Prop. 2.2]:

Lemma 7.1.5. *Let $u : (\mathbf{C}, \tau) \to (\mathbf{D}, \tau')$ be a functor. Let τ'_0 be a topology on $\mathbf{D}$ which is weaker than τ'. If u is τ'_0-locally cocontinuous, then the functor*

$$u_0^* : \mathcal{S}h_{\tau'_0}(\mathbf{D}) \hookrightarrow \mathcal{P}(\mathbf{D}) \xrightarrow{u^*} \mathcal{P}(\mathbf{C})$$

sends τ'-local equivalences to τ-local equivalences.

Proof. First of all, note that u_0^* admits a left adjoint

$$u_!^0 : \mathcal{P}(\mathbf{C}) \xrightarrow{u_!} \mathcal{P}(\mathbf{D}) \xrightarrow{\mathrm{L}_{\tau_0'}} \mathcal{S}h_{\tau_0'}(\mathbf{D})$$

by construction. Let $\mathrm{R}' \hookrightarrow h(d)$ be a τ'-covering sieve of an object d of $\mathbf{D}$. To show that $\vartheta : u_0^*(\mathrm{R}') \hookrightarrow u_0^*(h(d))$ is a τ-local equivalence, it suffices by universality of colimits to show that, for every object c of $\mathbf{C}$ and every morphism $\varphi : h(c) \to u_0^*h(d)$, the base change

$$u_0^*\mathrm{R}' \underset{u_0^*h(d)}{\times} h(c) \hookrightarrow h(c)$$

is a τ-covering sieve. Note that φ factors canonically through the unit morphism $h(c) \to u_0^*u_!^0h(c) = u_0^*h(u(c))$ and the canonical morphism $u_0^*u_!^0h(c) = u_0^*h(u(c)) \to u_0^*h(d)$ (obtained by adjunction from φ). The base change of ϑ by $u_0^*h(u(c)) \to u_0^*h(d)$ is identified, since u_0^* commutes with limits, with the canonical morphism

$$u^*(\mathrm{R}' \underset{h(d)}{\times} h(u(c))) \to u_0^*h(u(c)).$$

Since $\mathrm{R}' \times_{h(d)} h(u(c)) \hookrightarrow h(u(c))$ is τ'-covering, as the base change of a τ'-covering sieve, the conclusion follows by applying the condition (COC'). □

7.2. A topological digression, II: contractible colimits. In this paragraph we consider a topology whose associated category of sheaves coincides with the free completion by contractible colimits. This topology, which we denote cdm, is the one associated to the minimal cd-structure[10], with no commutative squares. We show that for any cdm-locally topologically cocontinuous functor u, the restriction functor u^* on sheaves commutes with contractible colimits.

7.2.1. Given an $(\infty, 1)$-category $\mathbf{C}$ admitting an initial object $\varnothing_{\mathbf{C}}$, the cdm topology is defined by the sieve $\varnothing \hookrightarrow h(\varnothing_{\mathbf{C}})$, where $\varnothing$ is the initial presheaf. A presheaf $\mathcal{F} : \mathbf{C}^{\mathrm{op}} \to \mathcal{S}pc$ is a cdm-sheaf if and only if the space $\mathcal{F}(\varnothing_{\mathbf{C}})$ is contractible. Let $\mathcal{S}h_{\mathrm{cdm}}(\mathbf{C})$ denote the $(\infty, 1)$-category of cdm-sheaves. It is not difficult to show that $\mathcal{S}h_{\mathrm{cdm}}(\mathbf{C})$ is the $(\infty, 1)$-category freely generated by $\mathbf{C}$ under contractible colimits.

Let $\mathrm{L}_{\mathrm{cdm}}$ denote the cdm-localization functor, left adjoint to the inclusion. For a presheaf $\mathcal{F}$ on $\mathbf{C}$, $\mathrm{L}_{\mathrm{cdm}}(\mathcal{F})$ can be described as the unique cdm-sheaf for which the space $\mathrm{L}_{\mathrm{cdm}}(\mathcal{F})(c)$ is identified with $\mathcal{F}(c)$ whenever c is not initial.

7.2.2. For convenience we state an easy-to-use sufficient condition for cdm-local cocontinuity.

Lemma 7.2.3. *Let $(\mathbf{C}, \tau)$ and $(\mathbf{D}, \tau')$ be ∞-sites and $u : \mathbf{C} \to \mathbf{D}$ a functor. Assume that $\mathbf{D}$ admits an initial object $\varnothing_{\mathbf{D}}$, and that the topology τ' is stronger than* cdm. *Then for the functor u to be* cdm*-locally cocontinuous, the following condition is sufficient:*

(COC'$_{\mathrm{cdm}}$) For every τ'-covering sieve $\mathrm{R}' \hookrightarrow h(u(c))$, the sieve $\mathrm{R} \hookrightarrow h(c)$, generated by morphisms $c' \to c$ such that either $h(u(c')) \to h(u(c))$ factors through $\mathrm{R}' \hookrightarrow h(u(c))$ or $u(c')$ is initial, is τ-covering.

Indeed let $c' \to c$ be a morphism such that $u(c')$ is initial. Then the (unique) morphism $h(u(c')) = \mathrm{L}_{\mathrm{cdm}}(\varnothing) \to h(u(c))$ factors as the composite of the (unique) morphism $\mathrm{L}_{\mathrm{cdm}}(\varnothing) \to \mathrm{L}_{\mathrm{cdm}}(\mathrm{R}')$ and $\mathrm{L}_{\mathrm{cdm}}(\mathrm{R}') \to h(u(c))$.

[10] See [Voe10] for the notion of cd-structure.

7.2.4. The following lemma is a formal consequence of [Chap. 1, Lemma 7.1.5], and the fact that cdm-sheaves are stable by contractible colimits:

Lemma 7.2.5. *Suppose that u is* cdm-*locally cocontinuous. Then the functor*

$$\mathcal{S}h_{\tau'}(\mathbf{D}) \xhookrightarrow{\mathrm{inc}_{\tau'}} \mathcal{P}(\mathbf{D}) \xrightarrow{u^*} \mathcal{P}(\mathbf{C}) \xrightarrow{\mathrm{L}_\tau} \mathcal{S}h_\tau(\mathbf{C})$$

commutes with contractible colimits.

Proof. Since the topology τ' is a refinement of the cdm topology, the inclusion $\mathrm{inc}_{\tau'}$ factors as

$$\mathrm{inc}_{\tau'} : \mathcal{S}h_{\tau'}(\mathbf{D}) \xhookrightarrow{\mathrm{inc}^{\mathrm{cdm}}_{\tau'}} \mathcal{S}h_{\mathrm{cdm}}(\mathbf{D}) \xhookrightarrow{\mathrm{inc}^{\mathrm{cdm}}} \mathcal{P}(\mathbf{D}).$$

Similarly the left adjoint $\mathrm{L}_{\tau'}$ factors as

$$\mathrm{L}_{\tau'} : \mathcal{P}(\mathbf{D}) \xrightarrow{\mathrm{L}^{\mathrm{cdm}}} \mathcal{S}h_{\tau'_{\mathrm{cdm}}}(\mathbf{D}) \xrightarrow{\mathrm{L}^{\mathrm{cdm}}_{\tau'}} \mathcal{S}h_{\tau'}(\mathbf{D}),$$

and $\mathrm{L}^{\mathrm{cdm}}_{\tau'}$ is left adjoint to $\mathrm{inc}^{\mathrm{cdm}}_{\tau'}$.

Given a diagram $(\mathcal{F}_i)_{i\in\mathbf{I}}$ of τ'-sheaves indexed by a contractible $(\infty,1)$-category $\mathbf{I}$, consider the counit morphism

$$\varinjlim_i \mathrm{inc}^{\mathrm{cdm}}_{\tau'}(\mathcal{F}_i) \longrightarrow \mathrm{inc}^{\mathrm{cdm}}_{\tau'} \mathrm{L}^{\mathrm{cdm}}_{\tau'} \varinjlim_i \mathrm{inc}^{\mathrm{cdm}}_{\tau'}(\mathcal{F}_i),$$

which is clearly a τ'-local equivalence. By applying $u^*_{\mathrm{cdm}} = u^* \mathrm{inc}_{\mathrm{cdm}}$ this induces a morphism

$$u^*_{\mathrm{cdm}} \varinjlim_i \mathrm{inc}^{\mathrm{cdm}}_{\tau'}(\mathcal{F}_i) \longrightarrow u^*_{\mathrm{cdm}} \mathrm{inc}^{\mathrm{cdm}}_{\tau'} \mathrm{L}^{\mathrm{cdm}}_{\tau'} \varinjlim_i \mathrm{inc}^{\mathrm{cdm}}_{\tau'}(\mathcal{F}_i),$$

which is identified with a canonical morphism

$$\varinjlim_i u^* \mathrm{inc}_{\tau'}(\mathcal{F}_i) \longrightarrow u^* \mathrm{inc}_\tau \varinjlim_i \mathcal{F}_i$$

since the inclusion $\mathrm{inc}_{\mathrm{cdm}}$ commutes with contractible colimits. By [Chap. 1, Lemma 7.1.5], this is a τ-local equivalence, so the claim follows. □

7.3. The exceptional inverse image functor $i^!$.

7.3.1. Let $i : \mathrm{Z} \hookrightarrow \mathrm{S}$ be a closed immersion. Note that if the base change functor $\mathcal{S}m_{/\mathrm{S}} \to \mathcal{S}m_{/\mathrm{Z}}$ were topologically cocontinuous (see [Chap. 1, Paragraph 7.1]), then the direct image functor i_* on Nisnevich sheaves would commute with arbitrary small colimits. Though this is not quite true, we will show that this is true cdm-*locally* (see [Chap. 1, Paragraph 7.2]), which will imply that i_* commutes with contractible colimits:

Proposition 7.3.2. *Let $i : \mathrm{Z} \hookrightarrow \mathrm{S}$ be a closed immersion. Then the direct image functor $i^{\mathcal{H}}_*$ commutes with contractible colimits.*

Proof. By [Chap. 1, Lemma 7.2.5] it suffices to show that the base change functor $\mathcal{S}m_{/\mathrm{S}} \to \mathcal{S}m_{/\mathrm{Z}}$ is cdm-locally cocontinuous. For this it suffices to check the condition (COC'$_{\mathrm{cdm}}$) of [Chap. 1, Lemma 7.2.3], which amounts to the following:

($*$) For any smooth S-scheme X and any Nisnevich covering sieve R' of $\mathrm{X_Z}$, the sieve R of X generated by morphisms $\mathrm{X}' \to \mathrm{X}$ such that either (i) the empty sieve on X'_Z is Nisnevich-covering, or (ii) $\mathrm{X}'_\mathrm{Z} \to \mathrm{X_Z}$ factors through R', is Nisnevich-covering.

This condition follows directly from [Chap. 0, Proposition 5.6.2], which says that étale morphisms can be lifted (Zariski-locally) along i. □

In particular:

Corollary 7.3.3. *The direct image functor $i^{\mathcal{H}\bullet}_*$ (resp. $i^{\mathcal{SH}}_*$) commutes with small colimits.*

By the adjoint functor theorem (see [Chap. 0, Paragraph 2.6.11]) we have a right adjoint $i^!_{\mathcal{H}_\bullet}$ (resp. $i^!_{\mathcal{SH}}$), called the *exceptional inverse image* functor.

7.4. The localization theorem. In this paragraph, we will work in the category of motivic spaces, and will omit the decoration $\mathcal{H}$ from the notation for simplicity.

7.4.1. Let $i : \mathrm{Z} \hookrightarrow \mathrm{S}$ be a closed immersion with quasi-compact open complement $j : \mathrm{U} \hookrightarrow \mathrm{S}$.

In this situation, we can construct a canonical commutative square

$$\begin{tikzcd} j_\sharp j^*(\mathcal{F}) \arrow[r] \arrow[d] & \mathcal{F} \arrow[d] \\ j_\sharp(e_{\mathrm{U}}) \arrow[r] & i_* i^*(\mathcal{F}). \end{tikzcd} \tag{7.3}$$

of motivic spaces over S; see [Chap. 2, Paragraph 3.3].

We call this the *localization square* associated to the pair (i, j).

7.4.2. The main theorem is this chapter is the following, due to [MV99] over classical bases (noetherian and of finite dimension).

Theorem 7.4.3. *Let $i : \mathrm{Z} \hookrightarrow \mathrm{S}$ be a closed immersion with quasi-compact open complement $j : \mathrm{U} \hookrightarrow \mathrm{S}$. Then for every motivic space $\mathcal{F}$ over* S*, the localization square (3.2) is cocartesian.*

The proof will occupy [Chap. 1, Sect. 9].

7.4.4. We can deduce from [Chap. 1, Theorem 7.4.3] a pointed version:

Corollary 7.4.5 (Localization)**.** *Let $i : \mathrm{Z} \hookrightarrow \mathrm{S}$ be a closed immersion, with quasi-compact open complement $j : \mathrm{U} \hookrightarrow \mathrm{S}$. For any pointed motivic space $(\mathcal{F}, s)$ over* S*, there is a canonical cofibre sequence*

$$j_\sharp j^*(\mathcal{F}, s) \to (\mathcal{F}, s) \to i_* i^*(\mathcal{F}, s). \tag{7.4}$$

and dually, a canonical fibre sequence

$$i_* i^!(\mathcal{F}, s) \to (\mathcal{F}, s) \to j_* j^*(\mathcal{F}, s) \tag{7.5}$$

of motivic spaces over S*.*

Proof. We want to show that the commutative square of pointed motivic spaces

$$\begin{tikzcd} j_\sharp j^*(\mathcal{F}, x) \arrow[r] \arrow[d] & (\mathcal{F}, x) \arrow[d] \\ \mathrm{pt}_{\mathrm{S}}^{\mathcal{S}pc_\bullet} \arrow[r] & i_* i^*(\mathcal{F}, x) \end{tikzcd}$$

is cocartesian.

Since the forgetful functor $(\mathcal{F}, x) \mapsto \mathcal{F}$ reflects contractible colimits ([Chap. 1, Lemma 3.2.3]), it suffices to show that the induced square of underlying motivic spaces

$$\begin{tikzcd} j_\sharp j^* \mathcal{F} \sqcup_{j_\sharp j^*(e_{\mathrm{S}})} e_{\mathrm{S}} \arrow[r] \arrow[d] & \mathcal{F} \arrow[d] \\ e_{\mathrm{S}}^{\mathcal{H}} \arrow[r] & i_* i^* \mathcal{F}. \end{tikzcd}$$

is cocartesian.

Consider the composite square

$$\begin{tikzcd} j_\sharp j^* \mathcal{F} \arrow[r] \arrow[d] & (j_\sharp j^* \mathcal{F}) \sqcup_{j_\sharp j^*(e_{\mathrm{S}})} e_{\mathrm{S}} \arrow[r] \arrow[d] & \mathcal{F} \arrow[d] \\ j_\sharp j^*(e_{\mathrm{S}}) \arrow[r] & e_{\mathrm{S}} \arrow[r] & i_* i^* \mathcal{F}. \end{tikzcd}$$

which is cocartesian by [Chap. 1, Theorem 7.4.3].

Since the left-hand square is evidently cocartesian, it follows that the right-hand square is also cocartesian. □

7.4.6. Similarly we also deduce localization for motivic spectra:

Corollary 7.4.7. *Let $i : \mathrm{Z} \hookrightarrow \mathrm{S}$ be a closed immersion, with quasi-compact open complement $j : \mathrm{U} \hookrightarrow \mathrm{S}$. For any motivic $\mathcal{T}$-spectrum $\mathbb{E}$ over S, there is a canonical cofibre sequence*

$$j_\sharp j^*(\mathbb{E}) \to \mathbb{E} \to i_* i^*(\mathbb{E}), \tag{7.6}$$

and dually a fibre sequence

$$i_* i^!(\mathbb{E}) \to \mathbb{E} \to j_* j^*(\mathbb{E}), \tag{7.7}$$

of motivic $\mathcal{T}$-spectra over S.

Proof. It suffices to show the first sequence is a cofibre sequence. Since the functors in question commute with small colimits, [Chap. 1, Proposition 4.2.4] allows us to the reduce to the case of pointed motivic spaces, which is [Chap. 1, Corollary 7.4.5]. □

7.4.8. An immediate corollary of [Chap. 1, Theorem 7.4.3] is:

Corollary 7.4.9. *Let $i : \mathrm{Z} \hookrightarrow \mathrm{S}$ be a closed immersion with quasi-compact open complement. Then the direct image functor $i_*^{\mathcal{H}}$ (resp. $i_*^{\mathcal{H}\bullet}$, $i_*^{\mathcal{SH}}$) is fully faithful.*

Proof. The claims for $i_*^{\mathcal{H}\bullet}$ and $i_*^{\mathcal{SH}}$ follow directly from that of $i_*^{\mathcal{H}}$.

Considering the localization square for $i_*(\mathcal{F})$, we see that the canonical morphism $i_* i^* i_* \to i_*$ is invertible. Hence it suffices to show that i_* is conservative.

For this, let $\varphi : \mathcal{F}_1 \to \mathcal{F}_2$ be a morphism of motivic spaces over Z such that $i_*(\varphi)$ is invertible. To show that φ is invertible, it suffices to show that

$$\Gamma(\mathrm{X}, \mathcal{F}_1) \to \Gamma(\mathrm{X}, \mathcal{F}_2)$$

is invertible for each smooth Z-scheme X.

By [Chap. 0, Proposition 5.6.2], we may assume that X is the base change of a smooth S-scheme Y. In this case the claim follows by assumption, since $\Gamma(\mathrm{X}, \mathcal{F}_i) = \Gamma(\mathrm{Y}, i_*(\mathcal{F}_i))$ for each i, by adjunction. □

7.5. Closed base change formula.

7.5.1. Let Θ be a cartesian square

$$\begin{tikzcd} \mathrm{X}_{\mathrm{Z}} \arrow[r, hook, "k"] \arrow[d, "g"] & \mathrm{X} \arrow[d, "f"] \\ \mathrm{Z} \arrow[r, hook, "i"] & \mathrm{S}, \end{tikzcd} \tag{7.8}$$

of schemes, with i and k closed immersions with quasi-compact open complements.

At the level of motivic spaces, there is a canonical 2-morphism

$$k_* g^* \to f^* i_* \tag{7.9}$$

constructed in [Chap. 2, Paragraph 2.3].

The following says that $\mathcal{H}$ satisfies the *right base change property* along closed immersions (see *loc. cit.*):

Corollary 7.5.2. *The 2-morphism (7.9) is invertible at the level of motivic spaces.*

Proof. This follows by considering the localization squares associated to the closed immersions j and k, respectively, and using the smooth base change formula ([Chap. 1, Proposition 6.2.2]). $\square$

7.5.3. In the pointed setting, the functor i_* admits a right adjoint $i^!$ ([Chap. 1, Corollary 7.3.3]), so we obtain another 2-morphism by right transposition from (7.9). Hence we have:

Corollary 7.5.4. *Given a cartesian square of the form (7.8), the canonical 2-morphisms*

$$k_* g^* \to f^* i_* \tag{7.10}$$

$$i^! f_* \to g_* k^! \tag{7.11}$$

are invertible at the level of pointed motivic spaces.

7.5.5. Fixing a family of pointed fibred spaces $(\mathcal{T}_{\mathrm{S}})_{\mathrm{S}}$ as in [Chap. 1, Paragraph 5.3], we have:

Corollary 7.5.6. *Given a cartesian square of the form (7.8), the canonical 2-morphisms*

$$k_* g^* \to f^* i_* \tag{7.12}$$

$$i^! f_* \to g_* k^! \tag{7.13}$$

are invertible at the level of motivic spectra.

7.6. Closed projection formula. Let $i : \mathrm{Z} \hookrightarrow \mathrm{S}$ be a closed immersion. Note that the symmetric monoidal functor $i^*_{\mathcal{S}pc}$ endows $\mathcal{H}(\mathrm{Z})$ with a structure of $\mathcal{H}(\mathrm{S})$-module category.

The following verifies the *right projection formula* along closed immersions, in the sense of [Chap. 2, Paragraph 2.3]:

Proposition 7.6.1. *The functor $i_*^{\mathcal{H}}$ lifts to a morphism of $\mathcal{H}(\mathrm{S})$-module categories. In other words, there are canonical isomorphisms*

$$i_*(\mathcal{G} \underset{\mathrm{Z}}{\times} i^*(\mathcal{F})) \to i_*(\mathcal{G}) \underset{\mathrm{S}}{\times} \mathcal{F} \tag{7.14}$$

for any motivic spaces $\mathcal{F}$ over S and $\mathcal{G}$ over Z, and dually

$$i^! \underline{\mathrm{Hom}}_{\mathrm{S}}(\mathcal{G}, \mathcal{F}) \to \underline{\mathrm{Hom}}_{\mathrm{Z}}(i^*\mathcal{G}, i^!\mathcal{F}) \tag{7.15}$$

for any motivic spaces $\mathcal{F}$ and $\mathcal{G}$ over S.

Proof. The second isomorphism is the right transpose of the first. The first follows from the localization theorem ([Chap. 1, Corollary 7.4.5]) and the smooth projection formula. $\square$

7.6.2. Similarly we get closed projection formulas for pointed motivic spaces and spectra. As above, the following statements are equivalent to formulas of the form (7.14) and (7.15). The proofs are completely analogous to those of [Chap. 1, Proposition 6.3.5] and [Chap. 1, Proposition 6.3.6].

Corollary 7.6.3. *The functor $i_*^{\mathcal{H}_\bullet}$ lifts to a morphism of $\mathcal{H}(\mathrm{S})_\bullet$-module categories.*

Fix a family of pointed fibred spaces $(\mathcal{T}_{\mathrm{S}})_{\mathrm{S}}$ as in [Chap. 1, Paragraph 5.3]. Then we have:

Corollary 7.6.4. *The functor $i_*^{\mathcal{SH}}$ lifts to a morphism of $\mathcal{SH}(\mathrm{S})$-module categories.*

7.7. Smooth-closed base change formula.

7.7.1. Let Θ be a cartesian square of schemes

$$\begin{array}{ccc} X_Z & \overset{k}{\hookrightarrow} & X \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ Z & \underset{i}{\hookrightarrow} & S, \end{array} \tag{7.16}$$

where i and j are closed immersions with quasi-compact open complements, and p and q are smooth.

There are canonical 2-morphisms

$$p_\sharp k_* \to i_* q_\sharp \tag{7.17}$$

$$i^! q^* \to k^! q^* \tag{7.18}$$

at the level of pointed motivic spaces, constructed in [Chap. 2, Paragraph 2.4].

The following verifies the *bidirectional base change property* with respect to smooth morphisms and closed immersions:

Corollary 7.7.2 (Smooth-closed base change)**.** *Given a cartesian square of the form (7.16), the* 2*-morphisms (7.17) and (7.18) are invertible at the level of pointed motivic spaces.*

Proof. The second transformation is obtained by passing to right adjoints from the first. For the first, it suffices by [Chap. 1, Corollary 7.4.9] it suffices to show that the transformation

$$p_\sharp k_* k^* \to i_* q_\sharp k^*,$$

obtained by pre-composition with k^*, is invertible. This follows directly from [Chap. 1, Corollary 7.4.5] and smooth base change. □

8. Thom spaces

8.1. Thom spaces.

8.1.1. Let E be a vector bundle over a scheme S. Denote by $p : \mathrm{E} \to \mathrm{S}$ the projection and by $s : \mathrm{S} \hookrightarrow \mathrm{E}$ the zero section.

We denote by $\mathrm{thom}_{\mathrm{E/S}}$ the *Thom suspension* endofunctor on the category of pointed motivic spaces, defined by the assignment

$$\mathcal{F} \mapsto \mathrm{thom}_{\mathrm{E/S}}(\mathcal{F}) := p_\sharp^{\mathcal{H}_\bullet} s_*^{\mathcal{H}_\bullet}(\mathcal{F}).$$

Its right adjoint $\mathrm{thom}^{\mathrm{E/S}}$, the *Thom desuspension* endofunctor, is given by

$$\mathcal{F} \mapsto \mathrm{thom}^{\mathrm{E/S}}(\mathcal{F}) := s^!_{\mathcal{H}_\bullet} p^*_{\mathcal{H}_\bullet}(\mathcal{F}).$$

The *Thom space* of E is the pointed motivic space

$$\mathrm{Th}_{\mathrm{S}}(\mathrm{E}) := \mathrm{thom}^{\mathrm{E/S}}(\mathbf{1}_{\mathrm{S}}^{\mathcal{H}_\bullet}).$$

8.1.2. We recall some results about Thom spaces that follow immediately from the various base change and projection formulas available to us. See [Chap. 2, Paragraph 3.4], where we will provide axiomatic proofs of these statements.

Lemma 8.1.3. *For each pointed motivic space $\mathcal{F}$ over* S*, there are canonical isomorphisms*

$$\mathrm{thom}_{\mathrm{E/S}}(\mathcal{F}) = \mathcal{F} \otimes_{\mathrm{S}} \mathrm{Th}_{\mathrm{S}}(\mathrm{E}), \tag{8.1}$$

$$\mathrm{thom}^{\mathrm{E/S}}(\mathcal{F}) = \underline{\mathrm{Hom}}_{\mathrm{S}}(\mathrm{Th}_{\mathrm{S}}(\mathrm{E}), \mathcal{F}). \tag{8.2}$$

Lemma 8.1.4. *For every vector bundle* E *over* S*, there are canonical isomorphisms*

$$f^* \circ \mathrm{thom}_{\mathrm{E/S}} \xrightarrow{\sim} \mathrm{thom}_{\mathrm{E}\times_{\mathrm{S}}\mathrm{T/T}} \circ f^*,$$

$$f_* \circ \mathrm{thom}^{\mathrm{E}\times_{\mathrm{S}}\mathrm{T/T}} \xrightarrow{\sim} \mathrm{thom}^{\mathrm{E/S}} \circ f_*.$$

In particular, for each pointed motivic space $\mathcal{F}$ over S*, there are canonical isomorphisms*

$$f^*(\mathrm{Th}_{\mathrm{S}}(\mathrm{E})) \xrightarrow{\sim} \mathrm{Th}_{\mathrm{T}}(\mathrm{E} \underset{\mathrm{S}}{\times} \mathrm{T}).$$

Lemma 8.1.5. *Let $p : \mathrm{X} \to \mathrm{S}$ be a smooth morphism. For each vector bundle* E *over* S*, there are canonical isomorphisms*

$$p_\sharp \circ \mathrm{thom}_{(\mathrm{E}\times_{\mathrm{S}}\mathrm{X})/\mathrm{X}} = \mathrm{thom}_{\mathrm{E/S}} \circ p_\sharp, \tag{8.3}$$

$$\mathrm{thom}^{\mathrm{E}\times_{\mathrm{S}}\mathrm{X/X}} \circ p^* = p^* \circ \mathrm{thom}^{\mathrm{E/S}}. \tag{8.4}$$

Lemma 8.1.6. *Let $i : \mathrm{Z} \hookrightarrow \mathrm{S}$ be a closed immersion. For any vector bundle* E *over* S*, there are canonical isomorphisms*

$$i_* \circ \mathrm{thom}_{(\mathrm{E}\times_{\mathrm{S}}\mathrm{Z})/\mathrm{Z}} = \mathrm{thom}_{\mathrm{E/S}} \circ i_*, \tag{8.5}$$

$$i^! \circ \mathrm{thom}^{\mathrm{E/S}} = \mathrm{thom}^{(\mathrm{E}\times_{\mathrm{S}}\mathrm{Z})/\mathrm{Z}} \circ i^!. \tag{8.6}$$

Lemma 8.1.7. *Let $\mathrm{E}' \to \mathrm{E} \to \mathrm{E}''$ be an exact sequence of vector bundles over* S*. Then there is a canonical* 2*-isomorphism*

$$\mathrm{thom}_{\mathrm{E/S}} \xrightarrow{\sim} \mathrm{thom}_{\mathrm{E''/S}} \circ \mathrm{thom}_{\mathrm{E'/S}}.$$

In particular, there is a canonical isomorphism of pointed motivic spaces

$$\mathrm{Th}_{\mathrm{S}}(\mathrm{E}) \xrightarrow{\sim} \mathrm{Th}_{\mathrm{S}}(\mathrm{E}'') \otimes_{\mathrm{S}} \mathrm{Th}_{\mathrm{S}}(\mathrm{E}'). \tag{8.7}$$

8.2. The Thom space of $\mathbf{A}^1$. The localization theorem allows us to give a more explicit description of the Thom space, which we will allow us to compute $\mathrm{Th}_{\mathrm{S}}(\mathbf{A}^1_{\mathrm{S}})$ as the motivic space represented by $\mathbf{P}^1_{\mathrm{S}}$.

8.2.1.

Lemma 8.2.2. *For any vector bundle* E *on* S*, there is a canonical isomorphism of pointed motivic spaces*[11]

$$\mathrm{Cofib}(\mathrm{M}_{\mathrm{S}}(\mathrm{E}^{\times}) \to \mathrm{M}_{\mathrm{S}}(\mathrm{E})) \xrightarrow{\sim} \mathrm{Th}_{\mathrm{S}}(\mathrm{E}),$$

where $\mathrm{E}^{\times}$ *denotes the open subscheme complementary to the zero section.*

Proof. Let $j : \mathrm{E}^{\times} \hookrightarrow \mathrm{E}$ denote the open immersion. Applying the cocontinuous functor $p_{\sharp}$ to the localization cofibre sequence associated to the closed immersion s, and evaluating the result on the motivic space $\mathbf{1}_{\mathrm{E}}$, we get the cofibre sequence

$$\mathrm{M}^{\bullet}_{\mathrm{S}}(\mathrm{E}^{\times}) \to \mathrm{M}^{\bullet}_{\mathrm{S}}(\mathrm{E}) \to \mathrm{Th}_{\mathrm{S}}(\mathrm{E})$$

of pointed motivic spaces over S. On underlying motivic spaces, this induces a cocartesian square

$$\begin{tikzcd}
\mathrm{M}_{\mathrm{S}}(\mathrm{E}^{\times}) \arrow[r] \arrow[d] & \mathrm{M}_{\mathrm{S}}(\mathrm{E}) \arrow[d] \\
\mathrm{M}_{\mathrm{S}}(\mathrm{E}^{\times}) \sqcup e_{\mathrm{S}} \arrow[r] \arrow[d] & \mathrm{M}_{\mathrm{S}}(\mathrm{E}) \sqcup e_{\mathrm{S}} \arrow[d] \\
e_{\mathrm{S}} \arrow[r] & \mathrm{Th}_{\mathrm{S}}(\mathrm{E})
\end{tikzcd}$$

as claimed. □

8.2.3. Let $\Sigma_{\mathbf{S}^1}$ denote the $\mathbf{S}^1$-suspension functor.

Corollary 8.2.4. *There are canonical isomorphisms*

$$(\mathrm{M}_{\mathrm{S}}(\mathbf{P}^1_{\mathrm{S}}), \infty) \xleftarrow{\sim} \Sigma_{\mathbf{S}^1}(\mathrm{M}_{\mathrm{S}}(\mathbf{A}^{1,\times}_{\mathrm{S}})) \xrightarrow{\sim} \mathrm{Th}_{\mathrm{S}}(\mathbf{A}^1_{\mathrm{S}}) \tag{8.8}$$

of pointed motivic spaces over S.

Proof. The right-hand isomorphism follows from [Chap. 1, Lemma 8.2.2] when we set $\mathrm{E} = \mathbf{A}^1_{\mathrm{S}}$, since $\mathrm{M}_{\mathrm{S}}(\mathbf{A}^1_{\mathrm{S}})$ is contractible. The left-hand isomorphism follows from Zariski descent by considering the standard open cover of $\mathbf{P}^1_{\mathrm{S}}$. □

We will often abuse notation and write simply $\mathbf{P}^1_{\mathrm{S}}$ for the pointed motivic space $(\mathrm{M}_{\mathrm{S}}(\mathbf{P}^1_{\mathrm{S}}), \infty)$ when there is no risk of confusion.

8.3. Motivic $\mathbf{P}^1$-spectra.

8.3.1. A *motivic spectrum* over a scheme S is a motivic $\mathbf{P}^1_{\mathrm{S}}$-spectrum[12], where we write $\mathbf{P}^1_{\mathrm{S}} = (\mathrm{M}_{\mathrm{S}}(\mathbf{P}^1_{\mathrm{S}}), \infty)$ by abuse of notation.

We will write $\mathcal{SH}(\mathrm{S}) := \mathcal{SH}_{\mathbf{P}^1_{\mathrm{S}}}(\mathrm{S})$ for the category of motivic $\mathbf{P}^1_{\mathrm{S}}$-spectra.

8.3.2.

Lemma 8.3.3. *The arena* $\mathcal{SH}(\mathrm{S})$ *is stable.*

Proof. It is clear that $\mathcal{SH}(\mathrm{S})$ is pointed. By [Chap. 1, Corollary 8.2.4], the 1-sphere $\mathbf{S}^1$ is invertible in $\mathcal{SH}(\mathrm{S})$, which is a necessary and sufficient condition for stability. □

[11]The cofibre is taken in the unpointed category, and we consider it with its canonical base point.

[12]This is the "correct" category of motivic spectra, where we will have the full formalism of the six operations. This is why we drop the subscript $\mathbf{P}^1_{\mathrm{S}}$ from the notation.

8.3.4. The following fact ensures that [Chap. 1, Lemma 4.1.10] applies, so that we get a canonical symmetric monoidal structure on $\mathcal{SH}(\mathrm{S})$.

Lemma 8.3.5. *The pointed motivic space $\mathbf{P}^1_{\mathrm{S}}$ is 3-symmetric.*

Proof. It suffices by functoriality to assume that $\mathrm{S} = \mathrm{Spec}(\mathbf{Z})$. In this case the claim is well-known (see e.g. [Voe98, Lem. 4.4]). ☐

8.3.6.

Lemma 8.3.7. *For each integer $n \geqslant 0$, there are canonical isomorphisms*

$$\mathrm{thom}_{\mathbf{A}^n_{\mathrm{S}}/\mathrm{S}} = \Sigma^n_{\mathbf{P}^1_{\mathrm{S}},\mathrm{S}},$$
$$\mathrm{thom}^{\mathbf{A}^n_{\mathrm{S}}/\mathrm{S}} = \Omega^n_{\mathbf{P}^1_{\mathrm{S}},\mathrm{S}}.$$

Proof. The second isomorphism is the right transpose of the first. For the first, note that by [Chap. 1, Lemma 8.1.7] and [Chap. 1, Corollary 8.2.4], both functors are given by the assignment

$$\mathcal{F} \mapsto \mathcal{F} \otimes (\mathbf{P}^1_{\mathrm{S}})^{\otimes n}.$$

☐

8.3.8. The adjunction $(\mathrm{thom}_{\mathrm{E/S}}, \mathrm{thom}^{\mathrm{E/S}})$ gives rise to an adjunction at the level of motivic spectra, which we denote in the same way.

We have (see [Chap. 2, Lemma 3.4.12]):

Lemma 8.3.9. *For every vector bundle E over S, the adjunction*

$$\mathrm{thom}_{\mathrm{E/S}} : \mathcal{SH}(\mathrm{S}) \rightleftarrows \mathcal{SH}(\mathrm{S}) : \mathrm{thom}^{\mathrm{E/S}} \tag{8.9}$$

is an equivalence.

8.3.10. Suppose the scheme S is classical, and noetherian of finite Krull dimension (in the classical sense). In this case there is an ordinary triangulated category of motivic spectra constructed by Morel–Voevodsky, which can be viewed as the underlying ordinary category of a stable $(\infty, 1)$-category (see [Rob14] or [Hoy15, Appendix C]).

We have:

Proposition 8.3.11. *If S is a classical noetherian scheme of finite Krull dimension, then the $(\infty, 1)$-category $\mathcal{SH}(\mathrm{S})$ coincides with the Morel–Voevodsky $(\infty, 1)$-category of motivic spectra over S.*

Proof. This follows directly from [Chap. 1, Proposition 2.4.14], by comparison with the construction of [Hoy15]. ☐

9. The localization theorem

This section is dedicated to the proof of the localization theorem (see [Chap. 1, Paragraph 7.4]).

Throughout the section, we let $i : \mathrm{Z} \hookrightarrow \mathrm{S}$ be a closed immersion of schemes, such that the complementary open immersion $j : \mathrm{U} \hookrightarrow \mathrm{S}$ is quasi-compact.

9.1. The space $\mathrm{h}_{\mathrm{S}}(\mathrm{X}, t)$.

9.1.1. Given a smooth S-scheme X, let $\mathrm{X_U} := \mathrm{X} \times_{\mathrm{S}} \mathrm{U}$ denote its base change along j, and $\mathrm{X_Z} := \mathrm{X} \times_{\mathrm{S}} \mathrm{Z}$ its base change along i.

We will write $\mathrm{h}_{\mathrm{S}}^{\mathrm{Z}}(\mathrm{X})$ for the space over S defined by the cocartesian square

$$\begin{tikzcd} \mathrm{h_S(X_U)} \arrow[r] \arrow[d] & \mathrm{h_S(X)} \arrow[d] \\ \mathrm{h_S(U)} \arrow[r] & \mathrm{h}_{\mathrm{S}}^{\mathrm{Z}}(\mathrm{X}). \end{tikzcd} \tag{9.1}$$

Note that there is a canonical isomorphism

$$i^*_{\mathcal{S}pc}(\mathrm{h}_{\mathrm{S}}^{\mathrm{Z}}(\mathrm{X})) = \mathrm{h_Z(X_Z)} \tag{9.2}$$

of spaces over Z, since $i^*_{\mathcal{S}pc}$ commutes with colimits.

Since colimits in $\mathcal{S}pc(\mathrm{S})$ are computed section-wise, we can describe the spaces of sections of $\mathrm{h}_{\mathrm{S}}^{\mathrm{Z}}(\mathrm{X})$ explicitly:

Lemma 9.1.2. *Let* Y *be a smooth* S*-scheme. If* $\mathrm{Y_Z}$ *is the empty scheme, then the space* $\Gamma(\mathrm{Y}, \mathrm{h}_{\mathrm{S}}^{\mathrm{Z}}(\mathrm{X}))$ *is contractible. Otherwise, there is a canonical isomorphism of spaces*

$$\Gamma(\mathrm{Y}, \mathrm{h}_{\mathrm{S}}^{\mathrm{Z}}(\mathrm{X})) = \Gamma(\mathrm{Y}, \mathrm{h_S(X)}) = \mathrm{Maps_S(Y, X)}. \tag{9.3}$$

9.1.3. Let $p : \mathrm{X} \to \mathrm{S}$ be a smooth morphism. Let $t : \mathrm{Z} \hookrightarrow \mathrm{X}$ be an S-morphism, i.e. a partially defined section of p.

Consider the canonical morphism

$$\varepsilon : \mathrm{h}_{\mathrm{S}}^{\mathrm{Z}}(\mathrm{X}) \to i_*^{\mathcal{S}pc} i^*_{\mathcal{S}pc}(\mathrm{h}_{\mathrm{S}}^{\mathrm{Z}}(\mathrm{X})) = i_*^{\mathcal{S}pc}(\mathrm{h_Z(X_Z)}) \tag{9.4}$$

induced by the counit of the adjunction $(i^*_{\mathcal{S}pc}, i_*^{\mathcal{S}pc})$.

The morphism t corresponds by adjunction to a morphism $\tau : \mathrm{h_S(S)} \to i_*^{\mathcal{S}pc}(\mathrm{h_S(X_Z)})$. We define a space $\mathrm{h_S}(\mathrm{X}, t)$ over S as the fibre of ε at the point τ, so that we have a cartesian square

$$\begin{tikzcd} \mathrm{h_S}(\mathrm{X}, t) \arrow[r] \arrow[d] & \mathrm{h}_{\mathrm{S}}^{\mathrm{Z}}(\mathrm{X}) \arrow[d, "\varepsilon"] \\ \mathrm{h_S(S)} \arrow[r, "\tau"] & i_*^{\mathcal{S}pc}(\mathrm{h_S(X_Z)}) \end{tikzcd}$$

of spaces over S.

9.1.4. Over S-schemes that do not vanish on Z, sections of $\mathrm{h_S}(\mathrm{X}, t)$ are S-sections of X extending t. More precisely (recall that limits in $\mathcal{S}pc(\mathrm{S})$ are computed section-wise):

Lemma 9.1.5. *Let* Y *be a smooth* S*-scheme. If* $\mathrm{Y_Z}$ *is the empty scheme, then the space* $\Gamma(\mathrm{Y}, \mathrm{h_S}(\mathrm{X}, t))$ *is contractible. Otherwise,* $\Gamma(\mathrm{Y}, \mathrm{h_S}(\mathrm{X}, t))$ *is canonically identified with the fibre of the restriction map*

$$\mathrm{Maps_S(Y, X)} \to \mathrm{Maps_Z(Y_Z, X_Z)}$$

at the point defined by the composite $\mathrm{Y_Z} \to \mathrm{Z} \overset{t}{\hookrightarrow} \mathrm{X_Z}$.

In other words, points of the space $\Gamma(\mathrm{Y}, \mathrm{h_S}(\mathrm{X}, t))$ are pairs (f, α), with $f : \mathrm{Y} \to \mathrm{X}$ an S-morphism and α a commutative triangle

$$\begin{tikzcd} \mathrm{Y_Z} \arrow[r, "f_{\mathrm{Z}}"] \arrow[d] & \mathrm{X_Z}. \\ \mathrm{Z} \arrow[ur, hook, "t"] & \end{tikzcd}$$

9.1.6. If p is a smooth morphism, then since $p^*_{\mathcal{S}pc}$ commutes with both limits and colimits, we have:

Lemma 9.1.7. *Let* X *be a smooth* S*-scheme and* $t : \mathrm{Z} \hookrightarrow \mathrm{X}$ *an* S*-morphism. If* $p : \mathrm{T} \to \mathrm{S}$ *is a smooth morphism, then there is a canonical isomorphism of fibred spaces*

$$p^*_{\mathcal{S}pc}(\mathrm{h_S}(\mathrm{X}, t)) = \mathrm{h_T}(\mathrm{X_T}, t_\mathrm{T}),$$

where $t_\mathrm{T} : \mathrm{Z_T} \hookrightarrow \mathrm{X_T}$ *is obtained from* t *by base change along* p*.*

9.1.8. Our main result about the fibred space $\mathrm{h_S}(\mathrm{X}, t)$ is as follows:

Proposition 9.1.9. *Let* X *be an affine smooth* S*-scheme. Then for every* S*-morphism* $t : \mathrm{Z} \hookrightarrow \mathrm{X}$*, the space* $\mathrm{h_S}(\mathrm{X}, t)$ *is motivically contractible.*

The proof will occupy the rest of this section.

9.1.10. We first consider the case of vector bundles:

Lemma 9.1.11. *Let* E *be a vector bundle over* S *with zero section* $s : \mathrm{S} \hookrightarrow \mathrm{E}$*. Then the space* $\mathrm{h_S}(\mathrm{E}, s_\mathrm{Z})$ *is motivically contractible, where* $s_\mathrm{Z} : \mathrm{Z} \hookrightarrow \mathrm{E_Z}$ *denotes the base change of* s *along* $i : \mathrm{Z} \hookrightarrow \mathrm{S}$*.*

Proof. It suffices to construct an $\mathbf{A}^1$-homotopy inverse to the unique morphism

$$\varphi : \mathrm{h_S}(\mathrm{E}, s_\mathrm{Z}) \to \mathrm{h_S}(\mathrm{S}).$$

The zero section induces a canonical morphism

$$\mathrm{h_S}(\mathrm{S}) \xrightarrow{s} \mathrm{h_S}(\mathrm{E}) \to \mathrm{h_S^Z}(\mathrm{E}),$$

which induces a canonical morphism

$$\psi : \mathrm{h_S}(\mathrm{S}) \to \mathrm{h_S}(\mathrm{E}, s_\mathrm{Z}).$$

It remains to define an $\mathbf{A}^1$-homotopy

$$\vartheta : \mathrm{h_S}(\mathbf{A}^1_\mathrm{S}) \underset{\mathrm{S}}{\times} \mathrm{h_S}(\mathrm{E}, s_\mathrm{Z}) \to \mathrm{h_S}(\mathrm{E}, s_\mathrm{Z})$$

between the identity and the composite $\psi \circ \varphi$. For each smooth S-scheme Y with $\mathrm{Y_Z} \neq \varnothing$, define

$$\Gamma(\mathrm{Y}, \vartheta) : \Gamma(\mathrm{Y}, \mathrm{h_S}(\mathbf{A}^1_\mathrm{S})) \times \Gamma(\mathrm{Y}, \mathrm{h_S}(\mathrm{E}, s_\mathrm{Z})) \to \Gamma(\mathrm{Y}, \mathrm{h_S}(\mathrm{E}, s_\mathrm{Z}))$$

by the assignment

$$(a : \mathrm{Y} \to \mathbf{A}^1_\mathrm{S}, f : \mathrm{Y} \to \mathrm{E}) \mapsto (a \cdot f : \mathrm{Y} \to \mathrm{E}).$$

It is clear that this defines the $\mathbf{A}^1$-homotopy desired. □

9.2. Étale base change.

9.2.1. The assignment $(\mathrm{X}, t) \mapsto \mathrm{h_S}(\mathrm{X}, t)$ is functorial in the following sense.

Let (X, t) and (X', t') be pairs, with X (resp. X′) a smooth S-scheme, and $t : \mathrm{Z} \hookrightarrow \mathrm{X}$ (resp. $t' : \mathrm{Z} \hookrightarrow \mathrm{X}'$) a partially defined section. Suppose $f : \mathrm{X}' \to \mathrm{X}$ is an S-morphism such that the square

$$\begin{array}{ccc} \mathrm{Z} & \xhookrightarrow{t'} & \mathrm{X}'_\mathrm{Z} \\ \| & & \downarrow \\ \mathrm{Z} & \xhookrightarrow{t} & \mathrm{X_Z} \end{array}$$

is cartesian. Then there is a canonical morphism of spaces over S

$$\mathrm{h_S}(\mathrm{X}', t') \to \mathrm{h_S}(\mathrm{X}, t). \tag{9.5}$$

Lemma 9.2.2. *Suppose that* (X, t) *and* (X', t') *are pairs as above. Let* $p : \mathrm{X}' \to \mathrm{X}$ *be an étale morphism, such that the above square is cartesian. Then the induced morphism*

$$\varphi : \mathrm{h}_{\mathrm{S}}(\mathrm{X}', t') \to \mathrm{h}_{\mathrm{S}}(\mathrm{X}, t)$$

is a Nisnevich-local equivalence.

The claim is that the induced morphism of Nisnevich sheaves $\mathrm{L}_{\mathrm{Nis}}(\varphi)$ is invertible. By [Chap. 0, Lemma 2.8.8], it suffices to show that it is 0-truncated (i.e. its diagonal is a monomorphism) and 0-connected (i.e. it is an effective epimorphism and so is its diagonal).

9.2.3. *Proof of [Chap. 1, Lemma 9.2.2], step 1.* To show that $\mathrm{L}_{\mathrm{Nis}}(\varphi)$ is 0-truncated, it suffices to show that φ is 0-truncated (since $\mathrm{L}_{\mathrm{Nis}}$ is exact). For this, it suffices to show that for each smooth S-scheme Y, the induced morphism of spaces of Y-sections

$$\Gamma(\mathrm{Y}, \varphi) : \Gamma(\mathrm{Y}, \mathrm{h}^{\mathrm{Z}}_{\mathrm{S}}(\mathrm{X}', t')) \to \Gamma(\mathrm{Y}, \mathrm{h}^{\mathrm{Z}}_{\mathrm{S}}(\mathrm{X}, t))$$

is 0-truncated.

We may assume Y_{Z} is not empty; then this is the morphism induced on fibres in the diagram

$$\begin{tikzcd}
\Gamma(\mathrm{Y}, \mathrm{h}^{\mathrm{Z}}_{\mathrm{S}}(\mathrm{X}', t')) \arrow[r] \arrow[d, dashed] & \mathrm{Maps}_{\mathrm{S}}(\mathrm{Y}, \mathrm{X}') \arrow[r] \arrow[d] & \mathrm{Maps}_{\mathrm{Z}}(\mathrm{Y}_{\mathrm{Z}}, \mathrm{X}'_{\mathrm{Z}}) \arrow[d] \\
\Gamma(\mathrm{Y}, \mathrm{h}^{\mathrm{Z}}_{\mathrm{S}}(\mathrm{X}, t)) \arrow[r] & \mathrm{Maps}_{\mathrm{S}}(\mathrm{Y}, \mathrm{X}) \arrow[r] & \mathrm{Maps}_{\mathrm{Z}}(\mathrm{Y}_{\mathrm{Z}}, \mathrm{X}_{\mathrm{Z}})
\end{tikzcd}$$

Note that the two right-hand vertical morphisms are 0-truncated: p is itself 0-truncated since it is étale, and since the Yoneda embedding commutes with limits, the induced morphism $\mathrm{h}_{\mathrm{S}}(\mathrm{X}') \to \mathrm{h}_{\mathrm{S}}(\mathrm{X})$ is also 0-truncated. It follows that the left-hand vertical morphism is also 0-truncated for each Y, and therefore so is φ.

9.2.4. *Proof of [Chap. 1, Lemma 9.2.2], step 2.* To show that $\mathrm{L}_{\mathrm{Nis}}(\varphi)$ is an effective epimorphism, it suffices to show that for each smooth S-scheme Y (with Y_{Z} not empty), any Y-section of $\mathrm{h}^{\mathrm{Z}}_{\mathrm{S}}(\mathrm{X}, t)$ can be lifted Nisnevich-locally along φ.

Let f be a Y-section of $\mathrm{h}^{\mathrm{Z}}_{\mathrm{S}}(\mathrm{X}, t)$, i.e. a morphism $f : \mathrm{Y} \to \mathrm{X}$ together with an isomorphism between f_{Z} and the composite $\mathrm{Y}_{\mathrm{Z}} \to \mathrm{Z} \xhookrightarrow{t} \mathrm{X}_{\mathrm{Z}}$. Let $q : \mathrm{Y}' \to \mathrm{Y}$ denote the base change of $p : \mathrm{X}' \to \mathrm{X}$ along f:

$$\begin{tikzcd}
\mathrm{Y}' \arrow[r, "q"] \arrow[d, "g"] & \mathrm{Y} \arrow[d, "f"] \\
\mathrm{X}' \arrow[r, "p"] & \mathrm{X}.
\end{tikzcd}$$

Then note that

$$\begin{tikzcd}
q^{-1}(\mathrm{Y}_{\mathrm{U}}) \arrow[r, hook] \arrow[d] & \mathrm{Y}' \arrow[d, "q"] \\
\mathrm{Y}_{\mathrm{U}} \arrow[r, hook] & \mathrm{Y}
\end{tikzcd}$$

is a Nisnevich square. Indeed, the closed immersion $\mathrm{Y}_{\mathrm{Z}} \hookrightarrow \mathrm{Y}$ is complementary to $\mathrm{Y}_{\mathrm{U}} \hookrightarrow \mathrm{Y}$, and it is clear that $q^{-1}(\mathrm{Y}_{\mathrm{Z}}) \to \mathrm{Y}_{\mathrm{Z}}$ is invertible because in the diagram

$$\begin{tikzcd}
q^{-1}(\mathrm{Y}_{\mathrm{Z}}) \arrow[r] \arrow[d] & \mathrm{Y}_{\mathrm{Z}} \arrow[d] \\
\mathrm{Z} \arrow[r, "\mathrm{id}_{\mathrm{Z}}"] \arrow[d, hook, "t"] & \mathrm{Z} \arrow[d, hook, "t'"] \\
\mathrm{X}'_{\mathrm{Z}} \arrow[r, "p_{\mathrm{Z}}"] & \mathrm{X}_{\mathrm{Z}}
\end{tikzcd}$$

the lower square and the composite square are cartesian, and hence so is the upper square.

Hence it suffices to show that the restriction of f to either component of this Nisnevich cover lifts to $\mathrm{h}_{\mathrm{S}}^{\mathrm{Z}}(\mathrm{X}', t')$. The restriction $f|\mathrm{Y}'$ lifts to a section of $\mathrm{h}_{\mathrm{S}}^{\mathrm{Z}}(\mathrm{X}', t')$ given by $g : \mathrm{Y}' \to \mathrm{X}'$. The restriction $f|\mathrm{Y}_{\mathrm{U}}$ admits a lift trivially: since $(\mathrm{Y}_{\mathrm{U}}) \times_{\mathrm{S}} \mathrm{Z} = \varnothing$, the spaces $\mathrm{h}_{\mathrm{S}}^{\mathrm{Z}}(\mathrm{X}, t)(\mathrm{Y}_{\mathrm{U}})$ and $\mathrm{h}_{\mathrm{S}}^{\mathrm{Z}}(\mathrm{X}', t')(\mathrm{Y}_{\mathrm{U}})$ are both contractible.

9.2.5. *Proof of [Chap. 1, Lemma 9.2.2], step 3.* It remains to show that the diagonal $\Delta_{\mathrm{L}_{\mathrm{Nis}}(\varphi)}$ of $\mathrm{L}_{\mathrm{Nis}}(\varphi)$ is an effective epimorphism, or equivalently that $\mathrm{L}_{\mathrm{Nis}}(\Delta_\varphi)$ is.

For each smooth S-scheme Y, the diagonal induces a morphism of spaces

$$\Gamma(\mathrm{Y}, \mathrm{h}_{\mathrm{S}}^{\mathrm{Z}}(\mathrm{X}', t')) \to \Gamma(\mathrm{Y}, \mathrm{h}_{\mathrm{S}}^{\mathrm{Z}}(\mathrm{X}', t')) \underset{\Gamma(\mathrm{Y}, \mathrm{h}_{\mathrm{S}}^{\mathrm{Z}}(\mathrm{X}, t))}{\times} \Gamma(\mathrm{Y}, \mathrm{h}_{\mathrm{S}}^{\mathrm{Z}}(\mathrm{X}', t')).$$

It suffices to show that for each Y (with Y_{Z} not empty), any Y-section of the target lifts Nisnevich-locally to a Y-section of the source. Choose a section of the target, given by two Y-sections $f : \mathrm{Y} \to \mathrm{X}'$ and $g : \mathrm{Y} \to \mathrm{X}'$, and an identification $\alpha : p \circ f \to p \circ g$.

Let $\mathrm{Y}_0 \hookrightarrow \mathrm{Y}$ denote the open immersion defined as the equalizer of the pair (f, g). Note that the closed immersion $\mathrm{Y}_{\mathrm{Z}} \hookrightarrow \mathrm{Y}$ factors through Y_0. Hence the open immersions $\mathrm{Y}_0 \hookrightarrow \mathrm{Y}$ and $\mathrm{Y}_{\mathrm{U}} \hookrightarrow \mathrm{Y}$ form a Zariski cover of Y. It is clear that the Y-section (f, g, α) lifts after restriction to Y_0 by definition, and after restriction to Y_{U} since $\mathrm{Y}_{\mathrm{U}} \times_{\mathrm{S}} \mathrm{Z} = \varnothing$, so the claim follows.

9.3. Reduction to the case of vector bundles. We reduce to the case of vector bundles in two steps: first, we show that partial sections of smooth S-schemes can be lifted Nisnevich-locally to globally defined sections; second, we show using étale base change that smooth S-schemes with globally defined sections can be replaced by their conormal bundles.

9.3.1. The following lemma will allow us to reduce to the situation where the Z-section t lifts to an S-section $s : \mathrm{S} \hookrightarrow \mathrm{X}$.

Lemma 9.3.2. *Let $p : \mathrm{X} \to \mathrm{S}$ be a smooth morphism. Given an S-morphism $t : \mathrm{Z} \hookrightarrow \mathrm{X}$, there exists a Nisnevich square*

(9.6)
$$\begin{tikzcd} \mathrm{Y}_{\mathrm{U}} \arrow[r, hook] \arrow[d] & \mathrm{Y} \arrow[d, "q"] \\ \mathrm{U} \arrow[r, hook, "j"] & \mathrm{S} \end{tikzcd}$$

such that q factors through p.

Proof. We will construct a commutative square

(9.7)
$$\begin{tikzcd} \mathrm{Z} \arrow[r, hook, dashed] \arrow[d, hook] & \mathrm{Y} \arrow[d, dashed] \\ \mathrm{X}_{\mathrm{Z}} \arrow[r, hook] & \mathrm{X}. \end{tikzcd}$$

with the following properties:

(i) The induced square of underlying classical schemes

$$\begin{tikzcd} \mathrm{Z}_{\mathrm{cl}} \arrow[r, hook] \arrow[d] & \mathrm{Y}_{\mathrm{cl}} \arrow[d] \\ (\mathrm{X}_{\mathrm{Z}})_{\mathrm{cl}} \arrow[r, hook] & \mathrm{X}_{\mathrm{cl}} \end{tikzcd}$$

is cartesian.

(ii) The composite morphism $\mathrm{Y} \to \mathrm{X} \to \mathrm{S}$ is étale.

Given such a square (9.7), it is clear that we get a Nisnevich square (9.6) as claimed, by taking q to be the composite $\mathrm{Y} \to \mathrm{X} \to \mathrm{S}$: indeed, the closed immersion $\mathrm{Z}_{\mathrm{cl}} \hookrightarrow \mathrm{S}$ is complementary to j, and the squares

$$\begin{tikzcd}
\mathrm{Z}_{\mathrm{cl}} \arrow[r, hook] \arrow[d] & \mathrm{Y}_{\mathrm{cl}} \arrow[r, hook] \arrow[d] & \mathrm{Y} \arrow[d, "q"] \\
\mathrm{Z}_{\mathrm{cl}} \arrow[r, hook] & \mathrm{S}_{\mathrm{cl}} \arrow[r, hook] & \mathrm{S}
\end{tikzcd}$$

are cartesian (the left-hand one by (i), the right-hand one by (ii)).

In the classical case, the existence of the square (9.7) is known (this is a non-equivariant version of [Hoy17, Thm. 2.21]).

Hence one obtains a cartesian square

$$\begin{tikzcd}
\mathrm{Z}_{\mathrm{cl}} \arrow[r, hook, dashed] \arrow[d, hook] & \mathrm{Y}_0 \arrow[d, dashed] \\
(\mathrm{X}_{\mathrm{Z}})_{\mathrm{cl}} \arrow[r, hook] & \mathrm{X}_{\mathrm{cl}}
\end{tikzcd}$$

of classical schemes. Then one defines Y by the cocartesian square of closed immersions

$$\begin{tikzcd}
\mathrm{Z}_{\mathrm{cl}} \arrow[r, hook] \arrow[d, hook] & \mathrm{Y}_0 \arrow[d, hook] \\
\mathrm{Z} \arrow[r, hook] & \mathrm{Y}.
\end{tikzcd}$$

By [GR16, III.1, Cor. 1.3.5] this is well-defined, and the morphism $\mathrm{Y}_0 \hookrightarrow \mathrm{Y}$ is a closed immersion identifying Y_0 with the classical scheme underlying Y. The existence of the desired commutative square

$$\begin{tikzcd}
\mathrm{Z} \arrow[r, hook] \arrow[d, hook] & \mathrm{Y} \arrow[d] \\
\mathrm{X}_{\mathrm{Z}} \arrow[r, hook] & \mathrm{X}
\end{tikzcd}$$

follows by construction. □

9.3.3. Next, we show that an S-section $s : \mathrm{S} \hookrightarrow \mathrm{X}$ may be approximated by the zero section of its conormal bundle. This is a refinement of [AG15, Prop. 2.1.10].

Lemma 9.3.4. *Let $p : \mathrm{X} \to \mathrm{S}$ be a smooth morphism, and assume that p is further affine. If p admits a section $s : \mathrm{S} \hookrightarrow \mathrm{X}$, then there exists an* S*-morphism $q : \mathrm{X} \to \mathbf{N}^*_s$ to the conormal bundle of s satisfying the following conditions:*

(i) The commutative diagram

$$\begin{tikzcd}
\mathrm{S} \arrow[r, hook, "s"] \arrow[d, equal] & \mathrm{X} \arrow[d, "q"] \\
\mathrm{S} \arrow[r, hook, "t"] & \mathbf{N}^*_s
\end{tikzcd}$$

is cartesian, where t denotes the zero section.

(ii) The morphism s factors through an open immersion $j_0 : \mathrm{X}_0 \hookrightarrow \mathrm{X}$ with $q \circ j_0$ étale.

Proof. Recall that the conormal bundle $\mathbf{N}^*_s$ is by definition the vector bundle associated to the shifted cotangent sheaf $\mathcal{N}^*_s = \mathcal{T}^*(\mathrm{S}/\mathrm{X})[-1]$.

Consider the closed immersion $s_{\mathrm{cl}} : \mathrm{S}_{\mathrm{cl}} \hookrightarrow \mathrm{X}_{\mathrm{cl}}$ of underlying classical schemes. Let $\mathcal{I}$ denote its defining quasi-coherent sheaf of ideals, and $\mathcal{N}_{s_{\mathrm{cl}}} = i^*(\mathcal{N}_s)$ its conormal bundle, where i is the closed immersion $\mathrm{S}_{\mathrm{cl}} \hookrightarrow \mathrm{S}$.

The epimorphism $(p_{\mathrm{cl}})_*(\mathcal{I}) \to \mathcal{N}^*_{s_{\mathrm{cl}}}$ admits a section, since $\mathcal{N}^*_{s_{\mathrm{cl}}}$ is projective, so that one obtains a morphism $\mathcal{N}^*_{s_{\mathrm{cl}}} \to (p_{\mathrm{cl}})_*(\mathcal{O}_{\mathrm{X}_{\mathrm{cl}}})$. This lifts to a morphism $\mathcal{N}^*_s \to p_*(\mathcal{O}_{\mathrm{X}})$, corresponding to a morphism of $\mathcal{O}_{\mathrm{S}}$-algebras

$$\varphi : \mathrm{Sym}_{\mathcal{O}_{\mathrm{S}}}(\mathcal{N}^*_s) \to p_*(\mathcal{O}_{\mathrm{X}}).$$

Then it is clear that the commutative square of $\mathcal{O}_{\mathrm{S}}$-algebras

$$\begin{tikzcd} \mathcal{N}^*_s \arrow[r, "\vartheta"] \arrow[d, "\varphi"] & \mathcal{O}_{\mathrm{S}} \arrow[d, equal] \\ p_*(\mathcal{O}_{\mathrm{X}}) \arrow[r, "\sigma"] & \mathcal{O}_{\mathrm{S}} \end{tikzcd}$$

is cocartesian.

We let $q : \mathrm{X} \to \mathbf{N}^*_s$ be the morphism of S-schemes corresponding to φ.

For (ii), let j_0 be the étale locus of q. To show that s factors through j_0, it is sufficient to note that $s^*(\mathcal{T}^*(\mathrm{X}/\mathbf{N}^*_s)) = 0$. $\square$

9.4. Motivic contractibility of $\mathrm{h}_{\mathrm{S}}(\mathrm{X}, t)$.

9.4.1. Let X be an affine smooth S-scheme, with structural morphism $p : \mathrm{X} \to \mathrm{S}$. Recall the statement of [Chap. 1, Proposition 9.1.9]: we want to show that for any S-morphism $t : \mathrm{Z} \hookrightarrow \mathrm{X}$, the fibred space $\mathrm{h}_{\mathrm{S}}(\mathrm{X}, t)$ is motivically contractible.

9.4.2. By [Chap. 1, Lemma 9.3.2] there exists a Nisnevich square

$$\begin{tikzcd} \mathrm{Y}_{\mathrm{U}} \arrow[r, hook] \arrow[d] & \mathrm{Y} \arrow[d, "q"] \\ \mathrm{U} \arrow[r, hook, "j"] & \mathrm{S} \end{tikzcd} \tag{9.8}$$

where q factors through $p : \mathrm{X} \to \mathrm{S}$. It suffices then by the Nisnevich separation property ([Chap. 1, Proposition 6.1.6]) to show that $j^* \mathrm{h}_{\mathrm{S}}(\mathrm{X}, t)$ and $q^* \mathrm{h}_{\mathrm{S}}(\mathrm{X}, t) = \mathrm{h}_{\mathrm{Y}}(\mathrm{Y} \times_{\mathrm{S}} \mathrm{X}, t')$ are contractible, where $t' : \mathrm{Y}_{\mathrm{Z}} \hookrightarrow (\mathrm{Y} \times_{\mathrm{S}} \mathrm{X})_{\mathrm{Z}}$ is the base change of t.

9.4.3. The case of $j^* \mathrm{h}_{\mathrm{S}}(\mathrm{X}, t)$ is clear, since j is complementary to $i : \mathrm{Z} \hookrightarrow \mathrm{S}$.

9.4.4. For $q^* \mathrm{h}_{\mathrm{S}}(\mathrm{X}, t)$, note that by construction there exists a section $t'' : \mathrm{Y} \hookrightarrow \mathrm{Y} \times_{\mathrm{S}} \mathrm{X}$ which lifts t' (since q factors through X):

$$\begin{tikzcd} (\mathrm{Y} \times_{\mathrm{S}} \mathrm{X})_{\mathrm{Z}} \arrow[r, hook] & \mathrm{Y} \times_{\mathrm{S}} \mathrm{X} \\ \mathrm{Y}_{\mathrm{Z}} \arrow[r, hook] \arrow[u, hook, "t'"] & \mathrm{Y} \arrow[u, hook, dashed, "t''"'] \end{tikzcd}$$

Hence by [Chap. 1, Lemma 9.3.4], [Chap. 1, Lemma 9.2.2] and [Chap. 1, Lemma 9.1.11], we have motivic equivalences

$$\mathrm{h}_{\mathrm{S}}(\mathrm{Y} \underset{\mathrm{S}}{\times} \mathrm{X}, t') = \mathrm{h}_{\mathrm{S}}(\mathbf{N}^*_{t''}, z) = \mathrm{h}_{\mathrm{S}}(\mathrm{S}),$$

where $\mathbf{N}^*_{t''}$ is the conormal bundle, and z is the base change of its zero section.

9.5. Proof of the localization theorem. We conclude this section by proving the localization theorem, using [Chap. 1, Proposition 9.1.9].

Recall that our goal is to show that the canonical morphism

$$\mathcal{F} \coprod_{j_\sharp j^*(\mathcal{F})} \mathrm{M}_\mathrm{S}(\mathrm{U}) \to i_* i^*(\mathcal{F}) \tag{9.9}$$

is invertible for each motivic space $\mathcal{F}$ over S.

9.5.1. First, note that we may reduce to the case where $\mathcal{F}$ is a motivic localization $\mathrm{M}_\mathrm{S}(\mathrm{X})$ of an affine smooth S-scheme X. Indeed, we have seen that the category $\mathcal{H}(\mathrm{S})$ is generated under sifted colimits by such objects ([Chap. 1, Proposition 2.4.12]) and that each of the functors $j_\sharp$, j^*, i^*, and i_* commutes with contractible colimits ([Chap. 1, Proposition 7.3.2]).

In this case the morphism (9.9) is canonically identified with the morphism

$$\mathrm{M}_\mathrm{S}(\mathrm{X}) \coprod_{\mathrm{M}_\mathrm{S}(\mathrm{X}_\mathrm{U})} \mathrm{M}_\mathrm{S}(\mathrm{U}) \to i_* \mathrm{M}_\mathrm{S}(\mathrm{X}_\mathrm{Z}) \tag{9.10}$$

where we write $\mathrm{X}_\mathrm{U} = \mathrm{X} \times_\mathrm{S} \mathrm{U}$ and $\mathrm{X}_\mathrm{Z} = \mathrm{X} \times_\mathrm{S} \mathrm{Z}$.

9.5.2. Note that the source of the morphism (9.10) is the motivic localization of the space $\mathrm{h}_\mathrm{S}^\mathrm{Z}(\mathrm{X})$, and that the target $i_*^{\mathcal{H}}(\mathrm{M}_\mathrm{Z}(\mathrm{X}_\mathrm{Z}))$ is the motivic localization of $i_*^{\mathcal{S}pc}(\mathrm{h}_\mathrm{Z}(\mathrm{X}_\mathrm{Z}))$.

Hence it suffices to show that the morphism

$$\mathrm{h}_\mathrm{S}^\mathrm{Z}(\mathrm{X}) \to i_*^{\mathcal{S}pc} \mathrm{h}_\mathrm{Z}(\mathrm{X}_\mathrm{Z}) \tag{9.11}$$

is a motivic equivalence.

9.5.3. By universality of colimits ([Chap. 1, Proposition 2.4.10]), it suffices to show that for every smooth S-scheme Y and every morphism $\mathrm{h}_\mathrm{S}(\mathrm{Y}) \to i_*^{\mathcal{S}pc} \mathrm{h}_\mathrm{Z}(\mathrm{X}_\mathrm{Z})$, corresponding to an S-morphism $t : \mathrm{Z} \to \mathrm{X}$, the base change

$$\mathrm{h}_\mathrm{S}^\mathrm{Z}(\mathrm{X}) \underset{i_*^{\mathcal{S}pc} \mathrm{h}_\mathrm{Z}(\mathrm{X}_\mathrm{Z})}{\times} \mathrm{h}_\mathrm{S}(\mathrm{Y}) \to \mathrm{h}_\mathrm{S}(\mathrm{Y}) \tag{9.12}$$

is invertible.

9.5.4. Let $p : \mathrm{Y} \to \mathrm{S}$ be the structural morphism of Y. Then since $\mathrm{h}_\mathrm{S}(\mathrm{Y}) = p_\sharp^{\mathcal{S}pc} \mathrm{h}_\mathrm{Y}(\mathrm{Y})$, one sees that (9.12) is identified, by the smooth projection formula ([Chap. 1, Lemma 6.3.3]), with a morphism

$$p_\sharp^{\mathcal{S}pc}(p_{\mathcal{S}pc}^* \mathrm{h}_\mathrm{S}^\mathrm{Z}(\mathrm{X}) \underset{p_{\mathcal{S}pc}^* i_*^{\mathcal{S}pc} \mathrm{h}_\mathrm{Z}(\mathrm{X}_\mathrm{Z})}{\times} \mathrm{h}_\mathrm{Y}(\mathrm{Y})) \to p_\sharp^{\mathcal{S}pc} \mathrm{h}_\mathrm{Y}(\mathrm{Y}). \tag{9.13}$$

9.5.5. Note that we have $p^* i_* = k_* q^*$ ([Chap. 1, Proposition 6.2.2]), where k (resp. q) is the base change of i (resp. p) along p (resp. i). Hence the morphism (9.13) is identified with the image by $p_\sharp$ of

$$\mathrm{h}_\mathrm{Y}^{\mathrm{Y}_\mathrm{Z}}(\mathrm{X} \underset{\mathrm{S}}{\times} \mathrm{Y}) \underset{k_*^{\mathcal{S}pc} \mathrm{h}_{\mathrm{Y}_\mathrm{Z}}((\mathrm{X} \times_\mathrm{S} \mathrm{Y})_\mathrm{Z})}{\times} \mathrm{h}_\mathrm{Y}(\mathrm{Y}) \to \mathrm{h}_\mathrm{Y}(\mathrm{Y}). \tag{9.14}$$

9.5.6. The source of the morphism (9.14) is nothing else than the space $\mathrm{h}_\mathrm{Y}(\mathrm{X} \times_\mathrm{S} \mathrm{Y}, t_\mathrm{Y})$, where $t_\mathrm{Y} : \mathrm{Z} \times_\mathrm{S} \mathrm{Y} \to \mathrm{X} \times_\mathrm{S} \mathrm{Y}$ is the base change of t along p. Hence we conclude by [Chap. 1, Proposition 9.1.9].

CHAPTER 2

The formalism of six operations

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1. Introduction

One of the main goals of this document is to construct the formalism of six operations for the stable motivic homotopy category, in the setting of derived algebraic geometry.

In this chapter, we develop an axiomatic approach to the construction of this formalism.

1.1. The six operations.

1.1.1. The principle of Grothendieck's yoga of six operations is that cohomology theories in algebraic geometry (e.g. ℓ-adic cohomology or de Rham cohomology), generally come with some *category of coefficients* (e.g. ℓ-adic sheaves or D-modules). These categories of coefficients come with the six operations $(f^*, f_*, f_!, f^!, \otimes, \underline{\mathrm{Hom}})$, which categorify the standard properties of cohomology theories like Künneth formulas, Poincaré duality and Gysin maps.

1.1.2. A category of coefficients satisfying the formalism of six operations consists roughly of the following data:

(1) For each (derived) scheme S, a closed symmetric monoidal $(\infty, 1)$-category $\mathbf{D}(\mathrm{S})$.

(2-a) For each morphism $f : \mathrm{S}' \to \mathrm{S}$, a functor of *inverse image*

$$f^* : \mathbf{D}(\mathrm{S}) \to \mathbf{D}(\mathrm{S}')$$

which is symmetric monoidal.

(2-b) For each morphism $f : \mathrm{S}' \to \mathrm{S}$, a functor of *direct image*

$$f_* : \mathbf{D}(\mathrm{S}') \to \mathbf{D}(\mathrm{S}),$$

right adjoint to f^*.

(3-a) For any morphism $f : \mathrm{S}' \to \mathrm{S}$ which is separated of finite type, a functor of *exceptional direct image* (or *direct image with compact support*)

$$f_! : \mathbf{D}(\mathrm{S}') \to \mathbf{D}(\mathrm{S}).$$

(3-b) For any morphism $f : \mathrm{S}' \to \mathrm{S}$ which is separated of finite type, a functor of *exceptional inverse image*

$$f^! : \mathbf{D}(\mathrm{S}) \to \mathbf{D}(\mathrm{S}'),$$

right adjoint to $f_!$.

(4) For any separated morphism of finite type f, a functorial 2-morphism

$$\alpha_f : f_! \to f_*$$

which is invertible when f is proper.

(5) For any cartesian square of schemes

$$\begin{tikzcd} \mathrm{T}' \arrow[r, "f'"] \arrow[d, "g'"] & \mathrm{S}' \arrow[d, "g"] \\ \mathrm{T} \arrow[r, "f"] & \mathrm{S} \end{tikzcd}$$

with g and g' separated of finite type, isomorphisms (*base change formulas*)

$$f^* g_! \xrightarrow{\sim} (g')_!(f')^*,$$

$$(f')_*(g')^! \xrightarrow{\sim} g^! f_*.$$

(6) For any morphism $f : \mathrm{S}' \to \mathrm{S}$ which is separated of finite type, isomorphisms (*projection formulas*)

$$\begin{aligned} \mathcal{F} \otimes_{\mathrm{S}} f_!(\mathcal{G}) &\xrightarrow{\sim} f_!(f^*(\mathcal{F}) \otimes_{\mathrm{S}'} \mathcal{G}), & (\mathcal{F} \in \mathbf{D}(\mathrm{S}), \mathcal{G} \in \mathbf{D}(\mathrm{S}')) \\ \underline{\mathrm{Hom}}_{\mathrm{S}'}(f^*(\mathcal{F}), f^!(\mathcal{G})) &\xrightarrow{\sim} f^! \underline{\mathrm{Hom}}_{\mathrm{S}}(\mathcal{F}, \mathcal{G}), & (\mathcal{F}, \mathcal{G} \in \mathbf{D}(\mathrm{S})) \\ f_* \underline{\mathrm{Hom}}_{\mathrm{S}'}(\mathcal{F}, f^!(\mathcal{G})) &\xrightarrow{\sim} \underline{\mathrm{Hom}}_{\mathrm{S}}(f_!(\mathcal{F}), \mathcal{G}). & (\mathcal{F} \in \mathbf{D}(\mathrm{S}'), \mathcal{G} \in \mathbf{D}(\mathrm{S})) \end{aligned}$$

1.1.3. All this data should come with a homotopy-coherent system of compatibilities.

For example, there should be connection isomorphisms $(g \circ f)^* \to f^* g^*$ and $(g \circ f)_! \to g_! f_!$ for any two composable morphisms of schemes f and g, with compatibilities between such isomorphisms.

The base change and projection formulas should also come with compatibilities with these connection isomorphisms.

1.2. The 2-category of correspondences.

1.2.1. As a starting point, we can coherently encode the individual operations and the adjunctions (f^*, f_*), $(f_!, f^!)$, $(\otimes, \underline{\mathrm{Hom}})$ by the data of a symmetric monoidal functor

$$\mathbf{D}^* : (\mathcal{S}ch)^{\mathrm{op}} \to \mathcal{A}rena,$$

and a functor

$$\mathbf{D}_! : \mathcal{S}ch \to \mathcal{A}rena$$

which take the same values on objects.

Here $\mathcal{A}rena$ denotes the $(\infty, 1)$-category of arenas (see [Chap. 0, Paragraph 2.6]).

We write $\mathbf{D}(\mathrm{S}) := \mathbf{D}^*(\mathrm{S}) = \mathbf{D}_!(\mathrm{S})$ for the $(\infty, 1)$-category associated to each scheme S. Since $\mathbf{D}^*$ is symmetric monoidal and preserves commutative monoid objects, there is an induced structure of commutative monoid in $\mathcal{A}rena$ on $\mathbf{D}(\mathrm{S})$ (since every scheme S has a canonical structure of cocommutative comonoid via the diagonal morphism). Recall that commutative monoids in $\mathcal{A}rena$ are by definition *symmetric monoidal arenas*, i.e. arenas with a symmetric monoidal structure such that the bifunctor $- \otimes -$ admits a right adjoint $\underline{\mathrm{Hom}}(-, -)$. Hence we have encoded the operations $(\otimes, \underline{\mathrm{Hom}})$.

For a morphism of schemes f, we write $f^* := \mathbf{D}^*(f)$ for the induced morphism of arenas, and f_* for its right adjoint. We have a functor

$$\mathbf{D}_* : \mathcal{S}ch \to (\infty, 1)\text{-}\mathcal{C}at$$

obtained from $\mathbf{D}^*$ by passing to right adjoints (see [Chap. 0, Paragraph 3.2]). Hence we have coherently encoded the operations (f^*, f_*).

Similarly, the operations $(f_!, f^!)$ are encoded by $\mathbf{D}_!$ and the functor

$$\mathbf{D}^! : (\mathcal{S}ch)^{\mathrm{op}} \to (\infty, 1)\text{-}\mathcal{C}at$$

obtained by passage to right adjoints.

1.2.2. In order to coherently encode base change formulas, we will follow the approach of D. Gaitsgory and N. Rozenblyum [GR16], using the $(\infty, 2)$-category of correspondences.

In [GR16, Book V.1] the authors define a 2-category $\mathbf{Corr}(\mathcal{S}ch)^{\mathrm{proper}}_{\mathrm{sep;all}}$, which can be described informally as follows. Its objects are schemes. The 1-morphisms $\mathrm{S}' \to \mathrm{S}$ are *correspondences*, i.e.

diagrams

$$
\begin{tikzcd}
\mathrm{T} \arrow[r, "f"] \arrow[d, "g"] & \mathrm{S}' \\
\mathrm{S}
\end{tikzcd}
\tag{1.1}
$$

where g is separated of finite type. The 2-morphisms from a correspondence (T', f', g') to a correspondence (T, f, g) are diagrams

$$
\begin{tikzcd}
\mathrm{T}' \arrow[dr, "h"] \arrow[drr, bend left, "f'"] \arrow[ddr, bend right, "g'"'] & & \\
& \mathrm{T} \arrow[r, "f"] \arrow[d, "g"] & \mathrm{S}' \\
& \mathrm{S} &
\end{tikzcd}
$$

where $h : \mathrm{T}' \to \mathrm{T}$ is proper.

Composition of 1-morphisms is given by forming fibred products, and composition of 2-morphisms is defined in the evident way.

1.2.3. The two functors $\mathbf{D}^*$ and $\mathbf{D}_!$, together with the base change formulas and all the relevant coherences, can be encoded by the datum of a symmetric monoidal functor

$$\mathbf{D}^*_! : \mathbf{Corr}(\mathcal{S}ch)^{\text{proper}}_{\text{sep;all}} \to (\mathbf{Arena})^{2\text{-op}}$$

on the 2-category of correspondences.

Given a scheme S, $\mathbf{D}^*_!$ sends it to a symmetric monoidal arena $\mathbf{D}(\mathrm{S})$. In particular, the closed symmetric monoidal structure encodes the operations $(\otimes, \underline{\mathrm{Hom}})$.

Given a correspondence of the form (1.1), $\mathbf{D}^*_!$ sends it to the composite

$$g_! f^* : \mathbf{D}(\mathrm{S}') \to \mathbf{D}(\mathrm{S}).$$

Hence by restricting $\mathbf{D}^*_!$ to the full sub-1-category of correspondences of the form (1.1) where the vertical morphisms are identity, we recover the functor $\mathbf{D}^*$ encoding the operations (f^*, f_*).

By restricting $\mathbf{D}^*_!$ to the full sub-1-category of correspondences where the horizontal morphisms are identity, we recover the functor $\mathbf{D}_!$ encoding the operations $(f_!, f^!)$.

1.2.4. Suppose there is a cartesian square of schemes

$$
\begin{tikzcd}
\mathrm{T}' \arrow[r, "f'"] \arrow[d, "g'"] & \mathrm{S}' \arrow[d, "g"] \\
\mathrm{T} \arrow[r, "f"] & \mathrm{S}
\end{tikzcd}
\tag{1.2}
$$

with g and g' separated of finite type.

The composite of the two correspondences

$$
\begin{tikzcd}
\mathrm{S}' \arrow[r, "\mathrm{id}"] \arrow[d, "g"] & \mathrm{S}' \\
\mathrm{S}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathrm{T} \arrow[r, "f"] \arrow[d, "\mathrm{id}"] & \mathrm{S} \\
\mathrm{T}
\end{tikzcd}
$$

is the correspondence

$$\begin{tikzcd}
\mathrm{T}' \arrow[r, "f'"] \arrow[d, "g'"] & \mathrm{S}' \arrow[r, "\mathrm{id}"] \arrow[d, "g"] & \mathrm{S}' \\
\mathrm{T} \arrow[r, "f"] \arrow[d, "\mathrm{id}"] & \mathrm{S} & \\
\mathrm{T} & &
\end{tikzcd}$$

Hence applying the functor $\mathbf{D}^*_!$, we obtain the base change isomorphism

$$f^* g_! \xrightarrow{\sim} (g')_!(f')^*.$$

1.2.5. Let $f : \mathrm{T} \to \mathrm{S}$ be a separated morphism of finite type. Then its diagonal $\Delta_{\mathrm{T}/\mathrm{S}} : \mathrm{T} \to \mathrm{T} \times_{\mathrm{S}} \mathrm{T}$ is a closed immersion (hence *a fortiori* proper), and therefore defines a morphism of correspondences $(\mathrm{T}, \mathrm{id}_{\mathrm{T}}, \mathrm{id}_{\mathrm{T}}) \to (\mathrm{T} \times_{\mathrm{S}} \mathrm{T}, \mathrm{pr}_1, \mathrm{pr}_2) = (\mathrm{T}, f, \mathrm{id}_{\mathrm{T}}) \circ (\mathrm{T}, \mathrm{id}_{\mathrm{T}}, f)$ given by the commutative diagram

$$\begin{tikzcd}
\mathrm{T} \arrow[dr, "\Delta_{\mathrm{T}/\mathrm{S}}"] \arrow[drr, equal, bend left] \arrow[ddr, equal, bend right] & & & \\
& \mathrm{T} \times_{\mathrm{S}} \mathrm{T} \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"] & \mathrm{T} \arrow[r, equal] \arrow[d, "f"] & \mathrm{T} \\
& \mathrm{T} \arrow[r, "f"] \arrow[d, equal] & \mathrm{S} & \\
& \mathrm{T} & &
\end{tikzcd}$$

Applying the 2-functor $\mathbf{D}^*_!$ (which is contravariant on 2-morphisms), we obtain a canonical functorial 2-morphism $\eta : f^* f_! \to \mathrm{id}_{\mathrm{T}}$, or by adjunction, a canonical functorial 2-morphism

$$\alpha_f : f_! \to f_*.$$

If f is further *proper*, then one has a morphism of correspondences $(\mathrm{T}, f, f) \to (\mathrm{S}, \mathrm{id}_{\mathrm{S}}, \mathrm{id}_{\mathrm{S}})$ given by the commutative diagram

$$\begin{tikzcd}
\mathrm{T} \arrow[dr, "f"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "f"'] & & \\
& \mathrm{S} \arrow[r, equal] \arrow[d, equal] & \mathrm{S} \\
& \mathrm{S} &
\end{tikzcd}$$

This gives rise to a canonical functorial 2-morphism $\varepsilon : \mathrm{id} \to f_! f^*$.

One checks that the 2-morphisms (η, ε) define an adjunction $(f^*, f_!)$. In particular, we see that the 2-morphism $\alpha_f : f_! \to f_*$ is invertible when f is proper.

1.2.6. Finally, the functor $\mathbf{Corr}(\mathcal{S}ch)^{\mathrm{proper}}_{\mathrm{sep;all}} \to (\mathbf{Arena})^{\text{2-op}}$ needs to be modified in order to encode the projection formulas. For this we make use of an observation implicit in [LZ12]: the projection formula

$$f_!(\mathcal{F} \otimes_{\mathrm{T}} f^*(\mathcal{G})) \xrightarrow{\sim} f_!(\mathcal{F}) \otimes_{\mathrm{S}} \mathcal{G}$$

is expressing nothing else than the $\mathbf{D}(\mathrm{S})$-*linearity* of the functor $f_! : \mathbf{D}(\mathrm{T}) \to \mathbf{D}(\mathrm{S})$. In other words, it expresses the fact that $f_!$ is a morphism of $\mathbf{D}(\mathrm{S})$-module categories, where $\mathbf{D}(\mathrm{T})$ is given the structure of $\mathbf{D}(\mathrm{S})$-module via the symmetric monoidal functor f^*.

With this in mind, we modify our functor $\mathbf{Corr}(\mathcal{S}ch)^{\mathrm{proper}}_{\mathrm{sep;all}} \to (\mathbf{Arena})^{\text{2-op}}$ as follows.

Firstly, we want to change the *target* from **Arena** to the 2-category **Arenamod** of pairs $(\mathbf{O}, \mathbf{C})$ with $\mathbf{O}$ a symmetric monoidal arena and $\mathbf{C}$ an $\mathbf{O}$-module arena. Morphisms $(\mathbf{O}, \mathbf{C}) \to (\mathbf{O}', \mathbf{C}')$ are given by pairs (u, v) with $u : \mathbf{O} \to \mathbf{O}'$ a symmetric monoidal morphism of arenas and $v : \mathbf{C} \to \mathbf{C}'$ a morphism of $\mathbf{O}$-module arenas, where $\mathbf{C}'$ is viewed as a $\mathbf{O}$-module via the functor u.

Given a morphism $f : \mathrm{T} \to \mathrm{S}$, we want to view f^* as a morphism

$$(\mathbf{D}(\mathrm{S}), \mathbf{D}(\mathrm{S})) \to (\mathbf{D}(\mathrm{S}), \mathbf{D}(\mathrm{T})) \tag{1.3}$$

and $f_!$ as a morphism

$$(\mathbf{D}(\mathrm{S}), \mathbf{D}(\mathrm{T})) \to (\mathbf{D}(\mathrm{S}), \mathbf{D}(\mathrm{S})). \tag{1.4}$$

For this, let $\mathcal{A}rrsch$ denote the category of *schematic arrows*, i.e. morphisms of schemes. Then we consider the functor

$$\mathbf{D}^*_{\mathcal{A}rrsch} : (\mathcal{A}rrsch)^{\mathrm{op}} \to \mathcal{A}renamod,$$

where $\mathcal{A}renamod$ denotes the underlying 1-category of **Arenamod**, defined on objects and morphisms as follows:

It sends an arrow $f : \mathrm{T} \to \mathrm{S}$ to the pair $(\mathbf{D}(\mathrm{S}), \mathbf{D}(\mathrm{T}))$, where $\mathbf{D}(\mathrm{T})$ is viewed as an $\mathbf{D}(\mathrm{S})$-module via the monoidal functor $f^* : \mathbf{D}(\mathrm{S}) \to \mathbf{D}(\mathrm{T})$.

It sends a morphism of arrows $\alpha : f' \to f$, given by a commutative square

$$\begin{tikzcd} \mathrm{T}' \arrow[r, "f'"] \arrow[d, "g'"] & \mathrm{S}' \arrow[d, "g"] \\ \mathrm{T} \arrow[r, "f"] & \mathrm{S}, \end{tikzcd}$$

to a morphism of pairs

$$(\mathbf{D}(\mathrm{S}), \mathbf{D}(\mathrm{T})) \to (\mathbf{D}(\mathrm{S}'), \mathbf{D}(\mathrm{T}'))$$

given by the monoidal functor

$$g^* : \mathbf{D}(\mathrm{S}) \to \mathbf{D}(\mathrm{S}')$$

and the $\mathbf{D}(\mathrm{S})$-linear functor

$$(g')^* : \mathbf{D}(\mathrm{T}) \to \mathbf{D}(\mathrm{T}').$$

This functor $\mathbf{D}^*_{\mathcal{A}rrsch}$ encodes the desired morphism (1.3) as the image of the morphism of schematic arrows $f \to \mathrm{id}_{\mathrm{S}}$ given by the commutative square

$$\begin{tikzcd} \mathrm{T} \arrow[r, "f"] \arrow[d, "f"] & \mathrm{S} \arrow[d, equal] \\ \mathrm{S} \arrow[r, equal] & \mathrm{S}. \end{tikzcd}$$

Similarly we want to encode the morphism (1.4) by a functor $\mathbf{D}_!^{\mathcal{A}rrsch} : \mathcal{A}rrsch \to \mathcal{A}renamod$.

1.2.7. Hence we need to change the *source* of our functor $\mathbf{D}_!^*$ to the 2-category

$$\mathbf{Corr}(\mathcal{A}rrsch)^{\mathrm{proper}}_{\mathrm{sep;all}},$$

of correspondences of *schematic arrows*, which can be defined in a similar way as $\mathbf{Corr}(\mathcal{S}ch)^{\mathrm{proper}}_{\mathrm{sep;all}}$.

In summary, the full formalism of six operations will be encoded by a symmetric monoidal 2-functor

$$\mathbf{Corr}(\mathcal{A}rrsch)^{\mathrm{proper}}_{\mathrm{sep;all}} \to (\mathbf{Arenamod})^{2\text{-op}}.$$

1.3. Biadjointable categories of coefficients.

1.3.1. We will refer to the datum of a symmetric monoidal functor

$$\mathbf{D}^* : (\mathcal{S}ch)^{\mathrm{op}} \to \mathcal{A}rena$$

simply as a *category of coefficients*.

We say that $\mathbf{D}^*$ is (open, proper)-*biadjointable* ([Chap. 2, Definition 2.4.4]) if the following conditions hold:

(1) $\mathbf{D}^*$ is *left-adjointable* along open immersions, i.e. for all open immersions j, the functor j^* admits a left adjoint $j_\sharp$ which satisfies base change and projection formulas.

(2) $\mathbf{D}^*$ is *right-adjointable* along proper morphisms, i.e. for all proper morphisms f, the functor f_* admits a right adjoint $f^!$, and f_* satisfies base change and projection formulas.

(3) $\mathbf{D}^*$ satisfies a base change formula expressing commutativity of the operations $j_\sharp$ and f_* (j an open immersion, f proper).

1.3.2. Given an (open, proper)-biadjointable category of coefficients, [Chap. 2, Theorem 4.2.2] says that $\mathbf{D}^*$ can be extended to a 2-functor

$$\mathbf{D}^*_! : \mathbf{Corr}(\mathcal{A}rrsch)^{\mathrm{proper}}_{\mathrm{sep;all}} \to (\mathbf{Arena})^{2\text{-op}}$$

encoding a formalism of six operations.

This functor is constructed by applying the technology of [GR16]. The basic idea, due to P. Deligne [SGA 4, Exp. XVII], is that the exceptional operations $(f_!, f^!)$ should be defined by choosing a compactification of f, i.e. a factorization of f as an open immersion j followed by a proper morphism g, and setting:

$$f_! = g_* j_!, \qquad f^! = j^* g^!.$$

More precisely, $f_!$ should be defined as the colimit of the contravariant functor $(j, p) \mapsto p_* j_!$ on the category of compactifications of f.

1.4. Motivic categories of coefficients.

1.4.1. In order to obtain the six functor formalism for the stable motivic homotopy category, we need to demonstrate (open, proper)-biadjointability.

What we have by construction is a category of coefficients which is left-adjointable with respect to *smooth* morphisms. That is, we have functors $p_\sharp$ left adjoint to p^* for all smooth morphisms p, satisfying base change and projection formulas.

[Chap. 2, Theorem 3.5.4] identifies a set of sufficient conditions for a category $\mathbf{D}^*$, left-adjointable along smooth, to be (open, proper)-biadjointable. We say that $\mathbf{D}^*$ is *motivic* ([Chap. 2, Definition 3.5.2]) if it satisfies these conditions. In particular, we can then apply [Chap. 2, Theorem 4.2.2] to obtain a full formalism of six operations on $\mathbf{D}^*$.

For the stable motivic homotopy category, these conditions have been verified in Chapter 1.

1.5. Organization of this chapter. In Sect. 2 we introduce the notion of *biadjointable category of coefficients*.

In Sect. 3 we introduce the property *motivic* for a category of coefficients, and show that any motivic category of coefficients is biadjointable. For classical schemes, the analogue of this result (in the language of triangulated categories) is due to J. Ayoub [Ayo07] (cf. [CD09]). We assume his result in our proof.

In Sect. 4 we show that any biadjointable category of coefficients extends to a functor on the category of correspondences, encoding the formalism of six operations. This result is a direct application of [GR16]. Our only departure from *loc. cit.* is the introduction of schematic arrows in order to encode projection formulas.

In Sect. 5 we return to our main example of interest, the category of motivic spectra. We construct this as a motivic category of coefficients and deduce the existence of the formalism of six operations.

2. Categories of coefficients

2.1. Categories of coefficients.

2.1.1. Recall that $\mathcal{A}rena$ denotes the symmetric monoidal $(\infty,1)$-category of arenas and colimit-preserving functors (see [Chap. 0, Paragraph 2.6]), and $\mathcal{A}renamon$ denotes the $(\infty,1)$-category of symmetric monoidal arenas, which are by definition commutative monoids in $\mathcal{A}rena$.

For the duration of this section, we fix an $(\infty,1)$-category $\mathbf{C}$ which admits fibred products. Note that the cartesian monoidal structure on $\mathbf{C}$ induces a canonical symmetric monoidal structure on $(\mathbf{C})^{\mathrm{op}}$.

Definition 2.1.2. *A* category of coefficients *(defined on* $\mathbf{C}$*) is a symmetric monoidal functor*

$$\mathbf{D}^* : (\mathbf{C})^{\mathrm{op}} \to \mathcal{A}rena.$$

Given a category of coefficients $\mathbf{D}^*$, we will write

$$\mathbf{D}(\mathrm{S}) := \mathbf{D}^*(\mathrm{S})$$

for the arena associated to an object $\mathrm{S} \in \mathbf{C}$. We will write $\varnothing_{\mathrm{S}}$ (resp. e_{S}) for the initial (resp. terminal) object of $\mathbf{D}(\mathrm{S})$. We will often refer to the objects of $\mathbf{D}(\mathrm{S})$ as *sheaves* on S.

For a morphism $f : \mathrm{T} \to \mathrm{S}$, we will write

$$f^* := \mathbf{D}^*(f) : \mathbf{D}(\mathrm{S}) \to \mathbf{D}(\mathrm{T})$$

for the induced functor, which we call the functor of *inverse image* along f. It is cocontinuous, and admits (by the adjoint functor theorem) a right adjoint

$$f_* : \mathbf{D}(\mathrm{T}) \to \mathbf{D}(\mathrm{S})$$

which we call the functor of *direct image* along f.

2.1.3. By passing to right adjoints (see [Chap. 0, Paragraph 3.2]), $\mathbf{D}^*$ gives rise to a unique functor

$$\mathbf{D}_* : \mathbf{C} \to (\infty,1)\text{-}\mathcal{C}at$$

such that each functor

$$\mathbf{D}_*(f) : \mathbf{D}(\mathrm{T}) \to \mathbf{D}(\mathrm{S})$$

is the right adjoint f_*.

2.1.4. Since the functor $\mathbf{D}^*$ underlying a category of coefficients is symmetric monoidal, it sends cocommutative comonoids in $\mathbf{C}$ to commutative monoids in $\mathcal{A}rena$. Note that every object $\mathrm{S} \in \mathbf{C}$ has a canonical structure of cocommutative comonoid (with respect to the cartesian monoidal structure).

Hence for each object $\mathrm{S} \in \mathbf{C}$, the arena $\mathbf{D}(\mathrm{S})$ has a canonical symmetric monoidal structure, and for each morphism f in $\mathbf{C}$, the inverse image functor f^* has a canonical symmetric monoidal structure (giving by adjunction a lax monoidal structure on its right adjoint f_*).

We will write $\otimes_{\mathrm{S}}$ for the monoidal product of $\mathbf{D}(\mathrm{S})$, $\mathbf{1}_{\mathrm{S}}$ for the monoidal unit, and $\underline{\mathrm{Hom}}_{\mathrm{S}}$ for the internal hom.

2.1.5. Dually, suppose we are given a functor (not necessarily symmetric monoidal)

$$\mathbf{D}_! : \mathbf{C} \to \mathcal{A}rena.$$

We will write $\mathbf{D}(\mathrm{S}) := \mathbf{D}_!(\mathrm{S})$ for the arena associated to an object $\mathrm{S} \in \mathbf{C}$. For each morphism $f : \mathrm{T} \to \mathrm{S}$ in $\mathbf{C}$, we write

$$f_! := \mathbf{D}_!(f) : \mathbf{D}(\mathrm{T}) \to \mathbf{D}(\mathrm{S})$$

for the induced functor, which commutes with colimits and admits a right adjoint $f^!$.

As above, we can pass to right adjoints to obtain a functor $\mathbf{D}^! : (\mathbf{C})^{\mathrm{op}} \to \mathcal{A}rena$.

2.1.6. For future use, we make the following definition:

Definition 2.1.7. *A category of coefficients* $\mathbf{D}^*$ *is* pointed *(resp.* $\mathbf{S}^1$-stable*,* compactly generated*) if the functor* $\mathbf{D}^* : (\mathbf{C})^{\mathrm{op}} \to \mathcal{A}rena$ *factors through the full subcategory spanned by pointed (resp. stable, compactly generated) arenas.*

2.2. Left-adjointability.

2.2.1. Let us fix a class *left* of *left-admissible* morphisms in $\mathbf{C}$, containing all isomorphisms, closed under composition and base change, and satisfying the 2-out-of-3 property. Let $\mathbf{C}^{left}$ denote the (non-full) subcategory of $\mathbf{C}$ spanned by left-admissible morphisms.

When $\mathbf{C}$ is the category of schemes, we will typically have *left* = open, the class of (quasi-compact) open immersions, or *left* = smooth, the class of smooth morphisms (of finite presentation).

Definition 2.2.2. *We say that the category of coefficients* $\mathbf{D}^*$ *is* weakly left-adjointable *along a morphism* $p : \mathrm{T} \to \mathrm{S}$ *if it satisfies the following property:*

(Adj^p) *The functor* p^* *admits a left adjoint* $p_\sharp$*.*

We say that $\mathbf{D}^*$ *is* weakly left-adjointable *along the class left if it satisfies the following property:*

(Adj^{left}) *For every left-admissible morphism p, the property* (Adj^p) *holds.*

Note that if $\mathbf{D}^*$ is weakly left-adjointable along *left*, then one obtains a canonical functor

$$\mathbf{D}_\sharp : \mathbf{C}^{left} \to \mathcal{A}rena$$

by passage to left adjoints (see [Chap. 0, Paragraph 3.2].

2.2.3. Recall the notion of adjointability of squares from [Chap. 0, Paragraph 3.3].

Definition 2.2.4. *The category of coefficients* $\mathbf{D}^*$ *satisfies* left base change *along a morphism* $p : \mathrm{S}' \to \mathrm{S}$ *if it is weakly left-adjointable along left, and the following property holds:*

(BC^p) *For all cartesian squares* Θ

$$\begin{tikzcd} \mathrm{T}' \arrow[r, "f'"] \arrow[d, "p'"] & \mathrm{S}' \arrow[d, "p"] \\ \mathrm{T} \arrow[r, "f"] & \mathrm{S}, \end{tikzcd} \tag{2.1}$$

the induced commutative square Θ^*

$$\begin{tikzcd} \mathbf{D}(\mathrm{S}) \arrow[r, "f^*"] \arrow[d, "p^*"] & \mathbf{D}(\mathrm{T}) \arrow[d, "(p')^*"] \\ \mathbf{D}(\mathrm{S}') \arrow[r, "(f')^*"] & \mathbf{D}(\mathrm{T}'). \end{tikzcd} \tag{2.2}$$

is vertically left-adjointable.

We say that $\mathbf{D}^*$ *satisfies* left base change *along the class left if it is weakly left-adjointable along left, and the following property holds:*

(BC^{left}) *For every left-admissible morphism p, the property* (BC^p) *holds.*

In other words, $\mathbf{D}^*$ satisfies left base change along a morphism p if for every such cartesian square Θ, the exchange 2-morphism

$$(p')_\sharp (f')^* \to f^* p_\sharp$$

is invertible.

2.2.5. For any morphism $f : \mathrm{T} \to \mathrm{S}$, the symmetric monoidal functor $f^* : \mathbf{D}(\mathrm{S}) \to \mathbf{D}(\mathrm{T})$ gives $\mathbf{D}(\mathrm{T})$ a structure of $\mathbf{D}(\mathrm{S})$-module category. If $f_\sharp$ is left adjoint to f^*, then by [Chap. 0, Lemma 2.7.7] it admits a canonical structure of colax morphism of $\mathbf{D}(\mathrm{S})$-modules. In particular there are canonical morphisms

$$f_\sharp(\mathcal{F} \otimes_\mathrm{T} f^*(\mathcal{G})) \to f_\sharp(\mathcal{F}) \otimes_\mathrm{S} \mathcal{G} \qquad (\mathcal{F} \in \mathbf{D}(\mathrm{T}), \mathcal{G} \in \mathbf{D}(\mathrm{S})).$$

Definition 2.2.6. *The category of coefficients* $\mathbf{D}^*$ *satisfies the* left projection formula *along a morphism* $p : \mathrm{T} \to \mathrm{S}$ *if it is weakly left-adjointable along p, and the following property holds:*

(Proj^p) *The colax morphism of* $\mathbf{D}(\mathrm{S})$*-modules* $p_\sharp$ *is* strict.

We say that $\mathbf{D}^*$ *satisfies the* left projection formula *along the class left if it is weakly left-adjointable along left, and the following property holds:*

(Proj^{left}) *For every left-admissible morphism p, the property* (Proj^{left}) *holds.*

In other words, $\mathbf{D}^*$ satisfies the left projection formula along $p : \mathrm{T} \to \mathrm{S}$ if the canonical morphisms

$$p_\sharp(\mathcal{F} \otimes_\mathrm{T} p^*(\mathcal{G})) \to p_\sharp(\mathcal{F}) \otimes_\mathrm{S} \mathcal{G} \qquad (\mathcal{F} \in \mathbf{D}(\mathrm{T}), \mathcal{G} \in \mathbf{D}(\mathrm{S})) \tag{2.3}$$

are invertible.

2.2.7.

Definition 2.2.8. *A category of coefficients* $\mathbf{D}^*$ *is* left-adjointable *along the class left if it is weakly left-adjointable along left* (Adj^{left})*, satisfies left base change along left* (BC^{left})*, and satisfies the left projection formula along left* (Proj^{left}).

2.3. Right-adjointability.

2.3.1. Let us fix a class *right* of *right-admissible* morphisms in $\mathbf{C}$, containing all isomorphisms, and closed under composition and base change. Let $\mathbf{C}^{right}$ denote the (non-full) subcategory of $\mathbf{C}$ spanned by right-admissible morphisms.

When $\mathbf{C}$ is the category of schemes, we will typically have *right* = closed, the class of closed immersions, or *right* = proper, the class of proper morphisms.

2.3.2. The following definitions are dual to the definitions in [Chap. 2, Paragraph 2.2].

Definition 2.3.3. *A category of coefficients* $\mathbf{D}^*$ *is* weakly right-adjointable *along a morphism q if it satisfies the following property:*

(Adj_q) *The direct image functor* q_* *admits a right adjoint.*

We say that $\mathbf{D}^*$ *is* weakly right-adjointable *along a class right if it satisfies the following property:*

(Adj_{right}) *For each right-admissible morphism q, the property* (Adj_q) *holds.*

Definition 2.3.4. *A category of coefficients $\mathbf{D}^*$ satisfies* right base change *along a morphism $q : \mathrm{S}' \to \mathrm{S}$ if the following property holds:*

(BC_{right}) *For all cartesian squares Θ in $\mathbf{C}$*

$$\begin{tikzcd} \mathrm{T}' \arrow[r, "f'"] \arrow[d, "q'"] & \mathrm{S}' \arrow[d, "q"] \\ \mathrm{T} \arrow[r, "f"] & \mathrm{S} \end{tikzcd}$$

with q and q' right-admissible, the induced commutative square Θ^*

$$\begin{tikzcd} \mathbf{D}(\mathrm{S}) \arrow[r, "f^*"] \arrow[d, "q^*"] & \mathbf{D}(\mathrm{T}) \arrow[d, "(q')^*"] \\ \mathbf{D}(\mathrm{S}') \arrow[r, "(f')^*"] & \mathbf{D}(\mathrm{T}'). \end{tikzcd}$$

is vertically right-adjointable.

We say that $\mathbf{D}^*$ satisfies right base change *along the class right if the following property holds:*

(BC_{right}) *For every right-admissible morphism q, the property* (BC_q) *holds.*

In other words, $\mathbf{D}^*$ satisfies right base change along a morphism q if for every such cartesian square Θ, the exchange 2-morphism

$$f^* p_* \to (p')_*(f')^*$$

is invertible.

Definition 2.3.5. *A category of coefficients $\mathbf{D}^*$ satisfies the* right projection formula *along a morphism $q : \mathrm{T} \to \mathrm{S}$ if the following property holds:*

(Proj_q) *The canonical structure of lax morphism of $\mathbf{D}(\mathrm{S})$-module arenas on q_* is* strict.

We say that $\mathbf{D}^*$ satisfies the right projection formula *along the class right if the following property holds:*

(Proj_{right}) *For every right-admissible morphism q, the property* (Proj_q) *holds.*

In other words, $\mathbf{D}^*$ satisfies the right projection formula along $q : \mathrm{T} \to \mathrm{S}$ if the canonical morphisms

$$q_*(\mathcal{F} \otimes_{\mathrm{T}} q^*(\mathcal{G})) \to q_*(\mathcal{F}) \otimes_{\mathrm{S}} \mathcal{G} \qquad (\mathcal{F} \in \mathbf{D}(\mathrm{T}), \mathcal{G} \in \mathbf{D}(\mathrm{S})). \tag{2.4}$$

are invertible.

Definition 2.3.6. *A category of coefficients $\mathbf{D}^*$ is* right-adjointable *along the class right if it is weakly right-adjointable along right* (Adj_{right})*, satisfies right base change along right* (BC_{right})*, and satisfies the right projection formula along right* (Proj_{right})*.*

2.4. Biadjointability.

2.4.1. Let $\mathbf{D}^*$ be a category of coefficients which is left-adjointable along *left*.

For every cartesian square Θ in $\mathbf{C}$

$$\begin{tikzcd} \mathrm{T}' \arrow[r, "q'"] \arrow[d, "p'"] & \mathrm{S}' \arrow[d, "p"] \\ \mathrm{T} \arrow[r, "q"] & \mathrm{S} \end{tikzcd} \tag{2.5}$$

with p and p' left-admissible, we have by left base change along *left* (BC^{left}) a commutative square

$$\begin{tikzcd} \mathbf{D}(\mathrm{S}) \arrow[r, "q^*"] & \mathbf{D}(\mathrm{T}) \\ \mathbf{D}(\mathrm{S}') \arrow[u, "p_\sharp"] \arrow[r, "(q')^*"] & \mathbf{D}(\mathrm{T}'). \arrow[u, "(p')_\sharp"'] \end{tikzcd} \tag{2.6}$$

Suppose that $\mathbf{D}^*$ is also right-adjointable along *right* and that q and q' are right-admissible. Then one can ask whether the above square is horizontally right-adjointable, i.e. whether the square

$$\begin{tikzcd} \mathbf{D}(\mathrm{S}) & \mathbf{D}(\mathrm{T}) \arrow[l, "q_*"] \\ \mathbf{D}(\mathrm{S}') \arrow[u, "p_\sharp"] & \mathbf{D}(\mathrm{T}') \arrow[l, "(q')_*"] \arrow[u, "(p')_\sharp"'] \end{tikzcd} \tag{2.7}$$

commutes via the 2-morphism

$$p_\sharp (q')_* \to q_* q^* p_\sharp (q')_* \simeq q_* (p')_\sharp (q')^* (q')_* \to q_* (p')_\sharp. \tag{2.8}$$

Definition 2.4.2. *The category of coefficients* $\mathbf{D}^*$ *satisfies* bidirectional base change *along the pair* (*left*, *right*) *if it satisfies left base change along left, right base change along right, and the following property holds:*

($\mathrm{BC}^{left}_{right}$) *For all cartesian squares* Θ *in* $\mathbf{C}$

$$\begin{tikzcd} \mathrm{T}' \arrow[r, "q'"] \arrow[d, "p'"] & \mathrm{S}' \arrow[d, "p"] \\ \mathrm{T} \arrow[r, "q"] & \mathrm{S} \end{tikzcd}$$

with p *and* p' *left-admissible (resp.* q *and* q' *right-admissible), the square (2.7) commutes.*

In other words, we require that for all cartesian squares Θ as above, the 2-morphism (2.8) is invertible.

2.4.3. Finally, we define:

Definition 2.4.4. *A category of coefficients* $\mathbf{D}^*$ *is* (*left*, *right*)-biadjointable *if it left-adjointable along left, right-adjointable along right, and satisfies bidirectional base change along* (*left*, *right*) ($\mathrm{BC}^{left}_{right}$).

In [Chap. 2, Sect. 4], we will show that on the category of schemes, any (open, proper)-biadjointable coefficient system can be extended to a full formalism of six operations, where open and proper denote the classes of open immersions and proper morphisms, respectively.

3. Motivic categories of coefficients

3.1. Premotivic categories of coefficients.

3.1.1. For the duration of this section, $\mathcal{S}ch$ will denote a full subcategory of the category of schemes, which is stable under coproducts and fibred products; the term *scheme* will refer to objects of $\mathcal{S}ch$. All categories of coefficients we consider will be defined on $\mathcal{S}ch$.

Let smooth (resp. open) denote the class of smooth morphisms of finite presentation (resp. of quasi-compact open immersions).

Definition 3.1.2. *A category of coefficients* $\mathbf{D}^*$ *is* premotivic *if it is left-adjointable along* smooth*.*

3.1.3. The following mild condition on $\mathbf{D}^*$ will always be satisfied in practice:

Definition 3.1.4. *A category of coefficients* $\mathbf{D}^*$ *is* additive *if, for any finite family of schemes* $(\mathrm{S}_\alpha)_\alpha$*, the canonical functor*

$$\mathbf{D}(\sqcup_\alpha \mathrm{S}_\alpha) \to \prod_\alpha \mathbf{D}(\mathrm{S}_\alpha)$$

is an equivalence.

In particular, we require that the category $\mathbf{D}(\varnothing)$ is trivial.

3.1.5. Let τ be a Grothendieck topology on the category $\mathcal{S}ch$.

Definition 3.1.6. *A category of coefficients* $\mathbf{D}^*$ *is* R-separated*, for a* τ*-covering sieve* R*, if it satisfies the following property:*

(Sep_R) *The family of functors* f^**, with* f *a morphism in* R*, is conservative.*

We say that $\mathbf{D}^*$ *is* τ-separated *if it satisfies the following property:*

(Sep_τ) *For every* τ*-covering sieve* R*,* $\mathbf{D}^*$ *satisfies the property* (Sep_R)*.*

Remark 3.1.7. Note that the sieve R is the τ-sieve generated by a family of morphisms $(f_i)_i$, then $\mathbf{D}^*$ is R-separated if and only if the family of functors $(f_i^*)_i$ is conservative.

In particular, if the topology τ is generated by a pre-topology τ_0, then $\mathbf{D}^*$ is τ-separated if and only if for every τ_0-covering family $(f_i)_i$, the family of functors $(f_i^*)_i$ is conservative.

3.2. Homotopy invariance.

3.2.1. Let $\mathbf{D}^*$ be a premotivic category of coefficients.

Definition 3.2.2. *A category of coefficients* $\mathbf{D}^*$ *is called* homotopy invariant *if it satisfies the following property:*

(Htp) *For every scheme* S*, and every vector bundle* $p : \mathrm{E} \to \mathrm{S}$*, the inverse image functor* $p^* : \mathbf{D}(\mathrm{S}) \to \mathbf{D}(\mathrm{E})$ *is fully faithful.*

The following observation is a basic property of adjunctions:

Lemma 3.2.3. *If* $\mathbf{D}^*$ *is premotivic, then the following conditions are equivalent:*

(i) $\mathbf{D}^*$ *is homotopy invariant.*

(ii) For every scheme S*, and every vector bundle* $p : \mathrm{E} \to \mathrm{S}$*, the counit morphism* $p_\sharp p^* \to \mathrm{id}$ *is invertible.*

The next lemma says that, in the presence of Zariski separation, it suffices to consider the projections $p : \mathbf{A}^1_{\mathrm{S}} \to \mathrm{S}$.

Lemma 3.2.4. *Let $\mathbf{D}^*$ be a premotivic category of coefficients. Consider the following conditions:*

(i) $\mathbf{D}^*$ is homotopy invariant.

(ii) For every scheme S and integer $n \geqslant 0$, the inverse image p^*, along the projection $p : \mathbf{A}^n_{\mathrm{S}} \to \mathrm{S}$, is fully faithful.

(iii) For every scheme S, the inverse image p^*, along the projection $p : \mathbf{A}^1_{\mathrm{S}} \to \mathrm{S}$, is fully faithful.

Then the conditions (ii) and (iii) are equivalent. If $\mathbf{D}^*$ satisfies the property of Zariski separation, then (i) and (ii) are also equivalent.

Proof. For $n \geqslant 0$, the projection $p : \mathbf{A}^n_{\mathrm{S}} \to \mathrm{S}$ can be written as a composite of n projections of the form $\mathbf{A}^1_{\mathrm{X}} \to \mathrm{X}$, for a scheme X. This demonstrates the equivalence between the conditions (ii) and (iii).

For the equivalence between (i) and (ii), assume that $\mathbf{D}^*$ satisfies Zariski separation. This property (together with the smooth base change formula) implies that the condition of the counit $p_\sharp p^* \to \mathrm{id}$ to be invertible, is Zariski-local in S. The conclusion follows. $\square$

3.3. Localization. In this paragraph we introduce the localization property, and restate some results of [CD09, §2.3] in our setting.

3.3.1. Let $i : \mathrm{Z} \hookrightarrow \mathrm{S}$ be a closed immersion with quasi-compact open complement $j : \mathrm{U} \hookrightarrow \mathrm{S}$. We deduce some immediate consequences of the base change property ($\mathrm{BC}^{\mathrm{smooth}}$) in this situation.

Considering the commutative square

$$\begin{tikzcd} \mathrm{U} \arrow[r, equal] \arrow[d, equal] & \mathrm{U} \arrow[d, hook, "j"] \\ \mathrm{U} \arrow[r, hook, "j"] & \mathrm{S}, \end{tikzcd}$$

which is cartesian because j is a monomorphism, we get:

Lemma 3.3.2. *Let $\mathbf{D}^*$ be a premotivic category of coefficients. Then for any quasi-compact open immersion $j : \mathrm{U} \hookrightarrow \mathrm{S}$, the canonical morphisms*

$$\mathrm{id} \to j^* j_\sharp,$$
$$j^* j_* \to \mathrm{id}$$

are invertible.

In other words, the functors $j_\sharp$ and j_* are fully faithful.

3.3.3. Consider the exchange 2-morphism (2.8)

$$\gamma_j : j_\sharp \to j_*. \tag{3.1}$$

associated to the cartesian square

$$\begin{tikzcd} \mathrm{U} \arrow[r, equal] \arrow[d, equal] & \mathrm{U} \arrow[d, hook] \\ \mathrm{U} \arrow[r, hook] & \mathrm{S}. \end{tikzcd}$$

We have:

Lemma 3.3.4. *Let $\mathbf{D}^*$ be an $\mathbf{S}^1$-stable category of coefficients. Let S be a scheme with connected components $(\mathrm{S}_\alpha)_\alpha$. Suppose that $\mathbf{D}(\varnothing) = 0$ and the canonical functor $\mathbf{D}(\mathrm{S}) \to \prod_\alpha \mathbf{D}(\mathrm{S}_\alpha)$ is conservative. Then the following conditions are true:*

(i) The canonical 2-morphism γ_{j_α} is invertible for each α, where j_α denotes the inclusion $\mathrm{S}_\alpha \hookrightarrow \mathrm{S}$.

(ii) The canonical functor $\mathbf{D}(\mathrm{S}) \to \prod_\alpha \mathbf{D}(\mathrm{S}_\alpha)$ is an equivalence.

Proof. For the first claim, it suffices by assumption to show that γ_{j_α} is invertible after applying any of the functors $(j_\beta)^*$. For $\alpha = \beta$, this follows from the fact that $(j_\alpha)_\sharp$ and $(j_\alpha)_*$ are fully faithful by [Chap. 2, Lemma 3.3.2]. For $\alpha \neq \beta$ this follows by left base change, using the assumption that $\mathbf{D}(\varnothing) = 0$.

For the second claim, we note that the functor in question admits a left adjoint

$$\prod_\alpha \mathbf{D}(\mathrm{S}_\alpha) \to \mathbf{D}(\mathrm{S})$$

given by the assignment $(\mathcal{F}_\alpha)_\alpha \mapsto \oplus_\alpha (j_\alpha)_\sharp(\mathcal{F}_\alpha)$. One checks that the unit 2-morphism is invertible, because $(j_\alpha)_\sharp$ are fully faithful ([Chap. 2, Lemma 3.3.2]), and the counit 2-morphism is invertible, which can be checked after application of each the functors $(j_\beta)^*$. □

In particular, we obtain:

Corollary 3.3.5. *If $\mathbf{D}^*$ is $\mathbf{S}^1$-stable and Zariski separated, then it is additive if and only if $\mathbf{D}(\varnothing) = 0$.*

3.3.6. Considering the cartesian square

$$\begin{tikzcd} \varnothing \arrow[r, hook] \arrow[d, hook] & \mathrm{Z} \arrow[d, hook, "i"] \\ \mathrm{U} \arrow[r, hook, "j"] & \mathrm{S}, \end{tikzcd}$$

we get:

Lemma 3.3.7. *Let $\mathbf{D}^*$ be a premotivic category of coefficients. Suppose that the category $\mathbf{D}(\varnothing)$ is trivial*[1]*. Then for any closed immersion $i : \mathrm{Z} \hookrightarrow \mathrm{S}$ with quasi-compact open complement $j : \mathrm{U} \hookrightarrow \mathrm{S}$, the canonical morphisms*

$$\varnothing_\mathrm{Z} \to i^* j_\sharp(\mathcal{F}) \qquad (\mathcal{F} \in \mathbf{D}(\mathrm{U})),$$
$$j^* i_*(\mathcal{F}) \to e_\mathrm{U} \qquad (\mathcal{F} \in \mathbf{D}(\mathrm{Z}))$$

are invertible.

3.3.8. Consider the canonical commutative square

$$\begin{tikzcd} j_\sharp j^*(\mathcal{F}) \arrow[r] \arrow[d] & \mathcal{F} \arrow[d] \\ j_\sharp j^* i_* i^*(\mathcal{F}) \arrow[r] & i_* i^*(\mathcal{F}) \end{tikzcd}$$

for any object $\mathcal{F} \in \mathbf{D}(\mathrm{S})$.

[1]By this we mean the terminal category, with a unique object and a unique morphism.

By [Chap. 2, Lemma 3.3.7] this induces a canonical commutative square

$$\begin{tikzcd} j_\sharp j^*(\mathcal{F}) \arrow[r] \arrow[d] & \mathcal{F} \arrow[d] \\ j_\sharp(e_{\mathrm{U}}) \arrow[r] & i_* i^*(\mathcal{F}) \end{tikzcd} \tag{3.2}$$

which we call the *localization square* associated to the pair (i, j).

Definition 3.3.9. *Let i be a closed immersion with quasi-compact open complement. The premotivic category of coefficients $\mathbf{D}^*$ satisfies the* localization property (Loc_i) *with respect to i, if the following conditions hold:*

(i) The category $\mathbf{D}(\varnothing)$ is trivial.

(ii) The functor i_* is fully faithful.

(iii) For every object $\mathcal{F} \in \mathbf{D}(\mathrm{S})$, the localization square (3.2) is cocartesian.

We say that $\mathbf{D}^*$ satisfies the localization property (Loc) *if it satisfies* (Loc_i) *for all closed immersions i with quasi-compact open complement.*

3.3.10. Let $\mathbf{D}^*$ be a premotivic category of coefficients satisfying $\mathbf{S}^1$-stability. In this case, condition (iii) of the localization property is equivalent to exactness of the triangle

$$j_\sharp j^*(\mathcal{F}) \to \mathcal{F} \to i_* i^*(\mathcal{F}) \qquad (\mathcal{F} \in \mathbf{D}(\mathrm{S})). \tag{3.3}$$

We also have the following reformulation:

Lemma 3.3.11. *Let $\mathbf{D}^*$ be a premotivic category of coefficients satisfying $\mathbf{S}^1$-stability. Then the following conditions are equivalent:*

(i) $\mathbf{D}^*$ satisfies the localization property.

(ii) The category $\mathbf{D}(\varnothing)$ is trivial, and for any closed immersion $i : \mathrm{Z} \hookrightarrow \mathrm{S}$ with quasi-compact open complement $j : \mathrm{U} \hookrightarrow \mathrm{S}$, the functor i_* is fully faithful and the pair (j^*, i^*) is conservative.

Proof. Suppose $\mathbf{D}^*$ satisfies the localization property. To show that (j^*, i^*) is conservative, it suffices by $\mathbf{S}^1$-stability to show that if $\mathcal{F}$ is a object of $\mathbf{D}(\mathrm{S})$ such that $j^*(\mathcal{F}) = 0$ and $i^*(\mathcal{F}) = 0$, then $\mathcal{F} = 0$. This follows immediately from the exactness of the triangle (3.3).

Conversely, suppose condition (ii) holds. It suffices to show that, for each object $\mathcal{F} \in \mathbf{D}(\mathrm{S})$, the canonical morphism φ in the commutative triangle

$$\begin{tikzcd} \mathcal{F} \arrow[rr, "\eta"] \arrow[dr, "\psi"'] & & i_* i^*(\mathcal{F}) \\ & \mathrm{Cofib}(j_\sharp j^*(\mathcal{F}) \to \mathcal{F}) \arrow[ur, "\varphi"'] & \end{tikzcd}$$

is invertible. By assumption, it suffices to show it it becomes invertible after applying either of the functors j^* or i^*.

Applying j^*, we get

$$j^* \mathrm{Cofib}(j_\sharp j^*(\mathcal{F}) \to \mathcal{F}) = 0$$

by [Chap. 2, Lemma 3.3.2], and

$$j^* i_* i^*(\mathcal{F}) = 0$$

by [Chap. 2, Lemma 3.3.7].

Applying i^*, we see that η becomes an isomorphism since i_* is fully faithful, and ψ becomes an isomorphism by [Chap. 2, Lemma 3.3.7]. Hence the conclusion follows. ∎

3.3.12. The localization property already implies left-adjointability along closed immersions and (smooth, closed)-base change, which is a big step towards (smooth, proper)-biadjointability:

Lemma 3.3.13. *Let* $\mathbf{D}^*$ *be a premotivic category of coefficients satisfying* $\mathbf{S}^1$*-stability and localization. Then* $\mathbf{D}^*$ *is* (smooth, closed)*-biadjointable.*

Proof. For weak left-adjointability ($\mathrm{Adj}_{\mathrm{closed}}$), let $i : \mathrm{Z} \hookrightarrow \mathrm{S}$ be a closed immersion and define a functor $i^! : \mathbf{D}(\mathrm{S}) \to \mathbf{D}(\mathrm{Z})$ by the formula

$$i^!(\mathcal{F}) := i^*(\mathrm{Fib}(\mathcal{F} \to j_*j^*(\mathcal{F}))) \qquad (\mathcal{F} \in \mathbf{D}(\mathrm{S})). \tag{3.4}$$

Write $\mathcal{K} = \mathrm{Fib}(\mathcal{F} \to j_*j^*(\mathcal{F}))$. By [Chap. 2, Lemma 3.3.2] we have $j^*(\mathcal{K}) = 0$. From the localization triangle it follows that $i_*i^!(\mathcal{F}) = i_*i^*(\mathcal{K}) = \mathcal{K}$. We define a unit 2-morphism $\eta : i_*i^! \to \mathrm{id}$ by the canonical morphism $i_*i^!(\mathcal{F}) = \mathcal{K} \to \mathcal{F}$.

By [Chap. 2, Lemma 3.3.7] we have $i^!i_*(\mathcal{F}) = i^*(\mathrm{Fib}(i_*\mathcal{F} \to j_*j^*i_*\mathcal{F})) = \mathcal{F}$. We define the counit $\varepsilon : \mathcal{F} \to i^!i_*(\mathcal{F}) = \mathcal{F}$ to be the identity.

It is straightforward to verify that η and ε verify the triangle identities defining an adjunction $(i_*, i^!)$.

For the properties ($\mathrm{BC}_{\mathrm{closed}}$), ($\mathrm{Proj}_{\mathrm{closed}}$), and ($\mathrm{BC}^{\mathrm{smooth}}_{\mathrm{closed}}$), the proof is exactly the same as in the case of motivic spectra; see Corollaries 7.5.6, 7.6.4, 7.7.2. □

In particular, by right transposition from the localization triangle (3.3)

$$j_\sharp j^*(\mathcal{F}) \to \mathcal{F} \to i_*i^*(\mathcal{F}) \qquad (\mathcal{F} \in \mathbf{D}(\mathrm{S})),$$

we obtain another exact triangle

$$i_*i^!(\mathcal{F}) \to \mathcal{F} \to j_*j^*(\mathcal{F}) \qquad (\mathcal{F} \in \mathbf{D}(\mathrm{S})). \tag{3.5}$$

3.3.14. Another indication of the strength of the localization property is the observation that it *forces* Nisnevich separation.

Lemma 3.3.15. *Let* $\mathbf{D}^*$ *be a premotivic category of coefficients satisfying* $\mathbf{S}^1$*-stability and localization. Then* $\mathbf{D}^*$ *satisfies the property of Nisnevich separation (and hence* a fortiori *Zariski separation).*

Proof. It suffices to show that for every Nisnevich square Q

$$\begin{tikzcd} \mathrm{V} \times_{\mathrm{S}} \mathrm{U} \arrow[r, hook] \arrow[d] & \mathrm{V} \arrow[d, "p"] \\ \mathrm{U} \arrow[r, hook, "j"] & \mathrm{S}, \end{tikzcd}$$

with p étale of finite presentation and j a quasi-compact open immersion, the pair of functors (j^*, p^*) is conservative.

By definition of Nisnevich square, there exists a closed immersion $i : \mathrm{Z} \hookrightarrow \mathrm{S}$ complementary to j such that in the cartesian square

$$\begin{tikzcd} \mathrm{V} \arrow[d, "p"] & \mathrm{V} \times_{\mathrm{S}} \mathrm{Z} \arrow[l, hook', "k"'] \arrow[d, "q"] \\ \mathrm{S} & \mathrm{Z} \arrow[l, hook', "i"] \end{tikzcd}$$

the morphism q is invertible.

By the localization property, the pair (j^*, i^*) is conservative. This is equivalent to the conservativity of the pair $(j^*, q^*i^*) = (j^*, k^*p^*)$, so the conclusion follows. □

3.3.16. Recall that a *nil-immersion* is a closed immersion that induces an isomorphism on underlying reduced classical schemes.

Definition 3.3.17. *A category of coefficients* $\mathbf{D}^*$ *satisfies* topological invariance *if for any nil-immersion* i*, the adjunction* (i^*, i_*) *is an equivalence.*

In particular, this property means that the category of coefficients $\mathbf{D}^*$ does not distinguish between a scheme S and its underlying classical scheme S_{cl} (or even its underlying reduced classical scheme $\mathrm{S}_{\mathrm{cl,red}}$).

Topological invariance follows automatically from the stronger property of localization:

Lemma 3.3.18. *Let* $\mathbf{D}^*$ *be a premotivic category of coefficients which is pointed and satisfies the localization property. Then* $\mathbf{D}^*$ *satisfies topological invariance.*

Proof. This follows immediately from the localization property applied to the closed immersion i, whose open complement is empty. □

3.4. Thom stability.

3.4.1. Let $\mathbf{D}^*$ be a premotivic category of coefficients. Let S be a scheme, $p : \mathrm{X} \to \mathrm{S}$ a smooth morphism of finite presentation, and $s : \mathrm{S} \hookrightarrow \mathrm{X}$ a section. Assume that p is separated, so that s is a closed immersion.

The *Thom suspension* endofunctor associated to the pair (p, s) is defined as

$$\mathrm{thom}_{p,s} := p_\sharp s_* : \mathbf{D}(\mathrm{S}) \to \mathbf{D}(\mathrm{S}).$$

If $\mathbf{D}^*$ is weakly right-adjointable with respect to closed immersions, then it admits a right adjoint, the *Thom desuspension* endofunctor

$$\mathrm{thom}^{p,s} := s^! p^* : \mathbf{D}(\mathrm{S}) \to \mathbf{D}(\mathrm{S}).$$

When there is no risk of confusion, we will also write $\mathrm{thom}_{\mathrm{X/S}} := \mathrm{thom}_{p,s}$ and $\mathrm{thom}^{\mathrm{X/S}} := \mathrm{thom}^{p,s}$ (when the latter is defined).

3.4.2. Let (p, s) be a pair as above and assume that p is étale. Then the closed immersion s is also an open immersion, i.e. it is an inclusion of a connected component. Assuming $\mathbf{D}^*$ is additive, we may apply [Chap. 2, Lemma 3.3.4] to deduce that there is a canonical 2-isomorphism $\mathrm{id} = p_\sharp s_\sharp \to \mathrm{thom}_{p,s}$. That is:

Lemma 3.4.3. *Let* $\mathbf{D}^*$ *be an additive category of coefficients. Let* p *be a separated* étale *morphism of finite presentation with a section* s*. Then the Thom suspension functor* $\mathrm{thom}_{p,s}$ *coincides with the identity endofunctor of* $\mathbf{D}(\mathrm{S})$*.*

3.4.4. Let $p : \mathrm{X} \to \mathrm{S}$ a smooth separated morphism of finite presentation, and $s : \mathrm{S} \hookrightarrow \mathrm{X}$ a section.

Given a morphism of schemes $f : \mathrm{T} \to \mathrm{S}$, consider the base change square

$$\begin{array}{ccc} \mathrm{Y} & \xrightarrow{g} & \mathrm{X} \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ \mathrm{T} & \xrightarrow{f} & \mathrm{S}. \end{array}$$

Let $t : \mathrm{T} \hookrightarrow \mathrm{Y}$ be the induced section of $q : \mathrm{Y} \to \mathrm{T}$.

Then we have:

Lemma 3.4.5. *Suppose that* $\mathbf{D}^*$ *is right-adjointable along closed immersions. Then there are canonical isomorphisms*

$$\mathrm{thom}_{q,t} \circ f^* = f^* \circ \mathrm{thom}_{p,s},$$
$$\mathrm{thom}^{p,s} \circ f_* = f_* \circ \mathrm{thom}^{q,t}.$$

Proof. The second isomorphism is the right transpose of the first. The first follows from left base change along smooth morphisms and right base change along closed immersions: we have $f^*p_\sharp s_* = q_\sharp g^* s_* = q_\sharp t_* f^*$. □

3.4.6. Suppose that there is a commutative diagram of schemes

(3.6)
$$\begin{tikzcd}
\mathrm{S} \arrow[d, hook, "t'"] \arrow[dr, hook, "s'"] & & \\
\mathrm{S}' \arrow[r, hook, "t"] \arrow[d, "q'"] & \mathrm{Y} \arrow[d, "q"] \arrow[dr, "p'"] & \\
\mathrm{S} \arrow[r, hook, "s"] & \mathrm{X} \arrow[r, "p"] & \mathrm{S}
\end{tikzcd}$$

where p (resp. p', q') is a smooth separated morphism with section s (resp. s', t').

If the square is cartesian, the exchange 2-morphism $q_\sharp t_* \to p_\sharp s_*$ ((2.8)) gives rise to a canonical 2-morphism $p_\sharp q_\sharp t_* t'_* \to p_\sharp s_* q'_\sharp t'_*$ by horizontal composition.

In other words, we have a canonical 2-morphism

$$\mathrm{thom}_{p',s'} \to \mathrm{thom}_{p,s} \circ \mathrm{thom}_{q',t'}. \tag{3.7}$$

By construction, we have:

Lemma 3.4.7. *If* $\mathbf{D}^*$ *satisfies bidirectional base change along* (smooth, closed)*, then the* 2*-morphism (3.7) is invertible.*

For example, this is true in the presence of localization ([Chap. 2, Lemma 3.3.13]).

By [Chap. 2, Lemma 3.4.3] we further deduce:

Corollary 3.4.8. *Assume that* $\mathbf{D}^*$ *satisfies the localization property. Suppose we have a commutative diagram as in (3.6). If q is étale, then there is a canonical* 2*-isomorphism*

$$\mathrm{thom}_{p',s'} \to \mathrm{thom}_{p,s}.$$

Proof. By 3.4.7 it suffices to show that the endofunctor $\mathrm{thom}_{q',t'}$ is the identity. Since q' is étale this follows from [Chap. 2, Lemma 3.4.3], which applies because Lemmas 3.3.15 and 3.3.5 ensure that $\mathbf{D}^*$ is additive. □

3.4.9. Suppose now that we have a commutative diagram

(3.8)
$$\begin{tikzcd}
\mathrm{S} \arrow[d, hook] \arrow[dr, hook] & & \\
\mathrm{E}' \arrow[r, hook, "t"] \arrow[d] & \mathrm{E} \arrow[d, "p"] \arrow[dr] & \\
\mathrm{S} \arrow[r, hook] & \mathrm{E}'' \arrow[r] & \mathrm{S}
\end{tikzcd}$$

where E, E′ and E″ are vector bundles over a scheme S, and the square is cartesian. (The unlabelled arrows $\hookrightarrow$ are the zero sections of the respective vector bundles.)

In this case the 2-morphism (3.7) takes the form

$$\mathrm{thom}_{\mathrm{E}/\mathrm{S}} \to \mathrm{thom}_{\mathrm{E}''/\mathrm{S}} \circ \mathrm{thom}_{\mathrm{E}'/\mathrm{S}}. \tag{3.9}$$

By 3.4.7 this is invertible when $\mathbf{D}^*$ satisfies bidirectional base change along the pair (smooth, closed).

3.4.10. We introduce the following property:

Definition 3.4.11. *The premotivic category of coefficients* $\mathbf{D}^*$ *is* Thom stable *if the following condition holds:*

(Thom) *For any pair* (p, s) *with* $p : \mathrm{X} \to \mathrm{S}$ *a smooth separated morphism and* $s : \mathrm{S} \to \mathrm{X}$ *a section, the Thom transformation* $\mathrm{thom}_{\mathrm{X}/\mathrm{S}}$ *is an equivalence.*

In practice, it suffices to consider the Thom transformation associated to the affine line:

Lemma 3.4.12. *Let* $\mathbf{D}^*$ *be a premotivic category of coefficients satisfying the localization property. Then the following conditions are equivalent:*

(i) $\mathbf{D}^*$ *is Thom stable.*

(ii) For every scheme S *and every vector bundle* $p : \mathrm{E} \to \mathrm{S}$*, the Thom transformation* $\mathrm{thom}_{\mathrm{E}/\mathrm{S}}$ *is an equivalence.*

(iii) For every scheme S*, the Thom transformation* $\mathrm{thom}_{\mathbf{A}^1_{\mathrm{S}}/\mathrm{S}}$ *is an equivalence.*

Proof. Recall that the localization property implies (smooth, closed)-biadjointability ([Chap. 2, Lemma 3.3.13]) and Zariski separation ([Chap. 2, Lemma 3.3.15]).

Let $p : \mathrm{X} \to \mathrm{S}$ be a separated smooth morphism of finite presentation with a section s. We begin by noting that the property of $\mathrm{thom}_{p,s}$ being an equivalence is Zariski-local on S. Indeed, by bidirectional base change along the pair (smooth, closed) we obtain canonical isomorphisms

$$j^* \circ \mathrm{thom}_{p,s} \xrightarrow{\sim} \mathrm{thom}_{p',s'},$$
$$\mathrm{thom}^{p',s'} \circ j^* \xrightarrow{\sim} j^* \circ \mathrm{thom}^{p,s}$$

where $p' : \mathrm{X} \times_{\mathrm{S}} \mathrm{U} \to \mathrm{U}$ is the base change of p and s' is the induced section of p'. Hence we conclude by Zariski separation.

Now we show that (iii) implies (ii). By the above observation, claim (ii) reduces to the statement that the functor $\mathrm{thom}_{\mathbf{A}^n_{\mathrm{S}}/\mathrm{S}}$ is an equivalence for each $n \geqslant 0$. But by (3.9) this functor is identified with the n-fold composite $(\mathrm{thom}_{\mathbf{A}^1_{\mathrm{S}}/\mathrm{S}})^{\circ n}$, so we conclude by assumption.

Next we show that (ii) implies (i). By [Chap. 1, Lemma 9.3.4], there exists a morphism $q : \mathrm{X} \to \mathbf{N}^*_{\mathrm{S}/\mathrm{X}}$ to the conormal bundle of s, which is étale on some open neighbourhood $\mathrm{U} \hookrightarrow \mathrm{X}$ of the image of s, such that the section s is the base change of the zero section of $\mathbf{N}^*_s$.

Let $\mathrm{S}_0 := \mathrm{S} \times_{\mathrm{X}} \mathrm{U}$ denote the base change of S to U, and $\mathrm{X}_0 := \mathrm{X} \times_{\mathrm{S}} \mathrm{S}_0$ the base change of X to S_0. Using [Chap. 2, Corollary 3.4.8] we obtain (arguing as in [CD09, Prop. 2.4.11]), a 2-isomorphism

$$\mathrm{thom}_{\mathrm{X}_0/\mathrm{S}_0} = \mathrm{thom}_{\mathbf{N}^*_{\mathrm{S}_0/\mathrm{X}_0}/\mathrm{S}_0}$$

and we conclude by (ii). □

3.4.13. Let $\mathbf{D}^*$ be a premotivic category of coefficients. Let $p : \mathrm{X} \to \mathrm{S}$ be a smooth separated morphism of finite presentation, and $s : \mathrm{S} \hookrightarrow \mathrm{X}$ a section.

Definition 3.4.14. *We define the* Thom object *associated to the pair* (p, s) *as the object*

$$\mathrm{Th}(p, s) := \mathrm{thom}_{p,s}(\mathbf{1}_{\mathrm{S}}) := p_\sharp s_*(\mathbf{1}_{\mathrm{S}})$$

in $\mathbf{D}(\mathrm{S})$.

If $p : \mathrm{E} \to \mathrm{S}$ is a vector bundle with zero section s, we write $\mathrm{Th}_{\mathrm{S}}(\mathrm{E}) := \mathrm{Th}(p, s)$ for the associated Thom object.

3.4.15. Using the projection formulas, one observes:

Lemma 3.4.16. *Suppose that $\mathbf{D}^*$ satisfies the right projection formula along closed immersions* $(\mathrm{Proj}^{\mathrm{closed}})$. *Let* S *be a scheme, $p : \mathrm{X} \to \mathrm{S}$ a smooth separated morphism, and $s : \mathrm{S} \hookrightarrow \mathrm{X}$ a section. Then we have canonical isomorphisms*

$$\mathrm{thom}_{p,s}(\mathcal{F}) = \mathrm{Th}(p, s) \otimes_{\mathrm{S}} \mathcal{F},$$
$$\mathrm{thom}^{p,s}(\mathcal{F}) = \underline{\mathrm{Hom}}_{\mathrm{S}}(\mathrm{Th}(p, s), \mathcal{F})$$

for each object $\mathcal{F} \in \mathbf{D}(\mathrm{S})$.

Proof. The second isomorphism is the right transpose of the first. For the first, we use the left projection formula along smooth morphisms and the right projection formula along closed immersions to write

$$p_\sharp s_*(\mathbf{1}_{\mathrm{S}}) \otimes \mathcal{F} = p_\sharp(s_*(\mathbf{1}_{\mathrm{S}}) \otimes p^*(\mathcal{F})) = p_\sharp(s_*(\mathbf{1}_{\mathrm{S}} \otimes_{\mathrm{S}} s^*p^*(\mathcal{F})) = p_\sharp s_* s^* p^*(\mathcal{F}) = p_\sharp s_*(\mathcal{F}),$$

as desired (since $p \circ s = \mathrm{id}$). □

3.4.17. Assume that we have a commutative diagram as in (3.8). In this case, the canonical 2-morphism (3.9) takes the form

$$\mathrm{Th}_{\mathrm{S}}(\mathrm{E}) \to \mathrm{Th}_{\mathrm{S}}(\mathrm{E}'') \otimes_{\mathrm{S}} \mathrm{Th}_{\mathrm{S}}(\mathrm{E}'). \tag{3.10}$$

Recall that this is invertible as soon as $\mathbf{D}^*$ satisfies bidirectional base change with respect to the pair $(\mathrm{smooth}, \mathrm{closed})$.

3.4.18. Putting the above together, we see that in practice, in order to force Thom stability it suffices to invert the Thom sheaves $\mathrm{Th}_{\mathrm{S}}(\mathbf{A}^1_{\mathrm{S}})$.

Corollary 3.4.19. *Let $\mathbf{D}^*$ be a premotivic category of coefficients satisfying the localization property. Then the following conditions are equivalent:*

(i) $\mathbf{D}^*$ is Thom stable.

(ii) For every scheme S*, every smooth separated morphism $p : \mathrm{X} \to \mathrm{S}$, and every section $s : \mathrm{S} \hookrightarrow \mathrm{X}$, the Thom object $\mathrm{Th}(p, s) \in \mathbf{D}(\mathrm{S})$ is invertible with respect to the monoidal product $\otimes_{\mathrm{S}}$.*

(iii) For every scheme S *and every vector bundle $p : \mathrm{E} \to \mathrm{S}$ with zero section $s : \mathrm{S} \hookrightarrow \mathrm{E}$, the Thom object $\mathrm{Th}_{\mathrm{S}}(\mathrm{E}) \in \mathbf{D}(\mathrm{S})$ is invertible with respect to the monoidal product $\otimes_{\mathrm{S}}$.*

(iv) For every scheme S*, the Thom object $\mathrm{Th}_{\mathrm{S}}(\mathbf{A}^1_{\mathrm{S}}) \in \mathbf{D}(\mathrm{S})$ is invertible with respect to the monoidal product $\otimes_{\mathrm{S}}$.*

Proof. By Lemmas 3.3.13 and 3.3.15, $\mathbf{D}^*$ satisfies the right projection formula along closed immersions, and Zariski separation. Hence we may apply Lemmas 3.4.12 and 3.4.16 to conclude. □

We deduce that, in the presence of localization, Thom stability is stronger than $\mathbf{S}^1$-stability.

Corollary 3.4.20. *Let $\mathbf{D}^*$ be a premotivic category of coefficients satisfying localization, Thom stability, and homotopy invariance. Then $\mathbf{D}^*$ is $\mathbf{S}^1$-stable.*

Proof. The proof is exactly as in the case of motivic spectra [Chap. 1, Lemma 8.3.3]. □

3.5. Motivic categories of coefficients.

3.5.1. The following is the natural analogue of the notion of *motivic triangulated category* of [CD09]:

Definition 3.5.2. *A premotivic category of coefficients* $\mathbf{D}^*$ *is* motivic *if it satisfies the properties of homotopy invariance* (Htp)*, Thom stability* (Thom)*, and localization* (Loc)*.*

3.5.3. Assume that all schemes in the category $\mathcal{S}ch$ are quasi-compact and quasi-separated.

In [Chap. 2, Paragraph 3.7] we will prove:

Theorem 3.5.4. *Let* $\mathbf{D}^*$ *be a compactly generated category of coefficients. If* $\mathbf{D}^*$ *is motivic, then it is* $(\mathrm{open}, \mathrm{proper})$*-biadjointable.*

3.6. Comparison with the axiomatic of Cisinski–Deglise.

3.6.1. Recall from [Lur16, Thm. 1.1.2.15] (cf. [GR16, Book-I.1, 5.1.2]) that for a stable $(\infty, 1)$-category $\mathbf{C}$, the underlying $(1, 1)$-category $(\mathbf{C})^{\mathrm{ordn}}$ admits a triangulated structure.

Let $\mathcal{A}renastab$ denote the full subcategory of $\mathcal{A}rena$ spanned by stable arenas. Let $\mathcal{C}attri^{\mathbf{V}}$ denote the (very large) $(2, 1)$-category of large triangulated categories, triangulated functors, and invertible triangulated natural transformations. We get:

Lemma 3.6.2. *The canonical functor of* $(\infty, 1)$*-categories* $\mathcal{A}renastab \to (1, 1)\text{-}\mathcal{C}at^{\mathbf{V}}$*, given on objects by the assignment* $\mathbf{C} \mapsto (\mathbf{C})^{\mathrm{ordn}}$*, lifts to a functor*

$$\mathcal{A}renastab \to \mathcal{C}attri^{\mathbf{V}} \tag{3.11}$$

along the forgetful functor $\mathcal{C}attri^{\mathbf{V}} \to (1, 1)\text{-}\mathcal{C}at^{\mathbf{V}}$*.*

Proof. Since the target of the desired functor is a $(2, 1)$-category, it is equivalent by adjunction to define a functor of $(2, 1)$-categories from the underlying $(2, 1)$-category of $\mathcal{A}renastab$, i.e. to define the functor on objects, 1-morphisms, and (invertible) 2-morphisms.

For this, it suffices to note that an exact functor of stable $(\infty, 1)$-categories induces a triangulated functor on triangulated categories, which follows directly upon inspection of the definition of the triangulated structure on $(\mathbf{C})^{\mathrm{ordn}}$. □

Further, when $\mathbf{C}$ is symmetric monoidal, the underlying $(1, 1)$-category $(\mathbf{C})^{\mathrm{ordn}}$ is *triangulated monoidal*. That is, it admits a symmetric monoidal structure which is compatible with the triangulated structure, in the sense that the monoidal product is exact and commutes with arbitrary coproducts in each argument. This follows from [Lur16, Rem. 2.1.2.20] and the fact that the monoidal product on $\mathbf{C}$ commutes with arbitrary small colimits in each argument.

Let $\mathcal{A}renamonstab$ denote the full subcategory of $\mathcal{A}renamon$ spanned by stable symmetric monoidal arenas. Let $\mathcal{C}attrimon^{\mathbf{V}}$ denote the (very large) $(2, 1)$-category of large triangulated monoidal categories, symmetric monoidal triangulated functors, and invertible symmetric monoidal triangulated natural transformations. We have:

Lemma 3.6.3. *The functor (3.11) lifts to a functor of* $(\infty, 1)$*-categories*

$$\mathcal{A}renamonstab \to \mathcal{C}attrimon^{\mathbf{V}} \tag{3.12}$$

along the forgetful functor $\mathcal{C}attrimon^{\mathbf{V}} \to \mathcal{C}attri^{\mathbf{V}}$*.*

3.6.4. Given a premotivic category of coefficients

$$\mathbf{D}^* : (\mathcal{S}ch)^{\mathrm{op}} \to \mathcal{A}renastab,$$

we can restrict to the $(1, 1)$-category of classical schemes and obtain a functor

$$(\mathbf{D}^*)^{\mathrm{cl}} : (\mathcal{S}ch^{\mathrm{cl}})^{\mathrm{op}} \to \mathcal{A}renastab,$$

which clearly defines a premotivic category of coefficients $(\mathbf{D}^*)^{\mathrm{cl}}$ on $\mathcal{S}ch^{\mathrm{cl}}$.

It is clear that, if $\mathbf{D}^*$ is motivic, then so is its restriction $(\mathbf{D}^*)^{\mathrm{cl}}$.

3.6.5. If $\mathbf{D}^*$ is $\mathbf{S}^1$-stable, then by post-composing $(\mathbf{D}^*)^{\mathrm{cl}}$ with the functor (3.11), we obtain a functor of $(2,1)$-categories

$$(\mathbf{D}^*)^{\mathrm{cl,tri}} : (\mathcal{S}ch^{\mathrm{cl}})^{\mathrm{op}} \to \mathcal{C}attri^{\mathbf{V}}$$

Since the functor $\mathbf{D}^* : (\mathcal{S}ch)^{\mathrm{op}} \to \mathcal{A}renastab$ is symmetric monoidal, $(\mathbf{D}^*)^{\mathrm{cl,tri}}$ factors through $\mathcal{C}attrimon^{\mathbf{V}}$ by [Chap. 2, Lemma 3.6.3].

It is immediate from the definitions that the functor $(\mathbf{D}^*)^{\mathrm{cl,tri}}$ defines a *premotivic triangulated category* in the sense of [CD09], which is *motivic* whenever $(\mathbf{D}^*)^{\mathrm{cl}}$ is. That is:

Lemma 3.6.6. *Let $\mathbf{D}^*$ be an $\mathbf{S}^1$-stable premotivic category of coefficients $\mathbf{D}^*$. Then the following hold:*

(i) The induced functor $(\mathbf{D}^*)^{\mathrm{cl,tri}}$ defines a premotivic triangulated category.

(ii) If $\mathbf{D}^*$ is a motivic category of coefficients, then the premotivic triangulated category $(\mathbf{D}^*)^{\mathrm{cl}}$ satisfies the localization, homotopy and stability properties (in the sense of loc. cit*).*

3.6.7. Conversely, the following lemma allows us to deduce properties of the motivic category of coefficients $(\mathbf{D}^*)^{\mathrm{cl}}$ from the corresponding properties of the motivic triangulated category $(\mathbf{D}^*)^{\mathrm{cl,tri}}$.

Lemma 3.6.8. *Let $\mathbf{D}^*$ be a motivic category of coefficients. Then the motivic category of coefficients $(\mathbf{D}^*)^{\mathrm{cl}}$ has weak right-adjointability (resp. right base change, the right projection formula, bidirectional base change) if and only if the premotivic triangulated category $(\mathbf{D}^*)^{\mathrm{cl,tri}}$ has the adjoint property (resp. the proper base change formula, the proper projection formula, the support property) in the sense of [CD09].*

Proof. The properties in question involve statements of the following two forms:

(1) A certain exact functor of stable $(\infty,1)$-categories is an equivalence.

(2) A certain exact functor of stable arenas commutes with small colimits (i.e. admits a right adjoint).

(3) A certain 2-morphism between functors of $(\infty,1)$-categories is invertible.

For statements of the first form it is sufficient to recall that an exact functor of stable $(\infty,1)$-categories is an equivalence if and only if it induces a triangulated equivalence of underlying $(1,1)$-categories.

For statements of the second form, let u be an exact functor of stable $(\infty,1)$-categories. For u to commute with small colimits, it suffices that it commutes with arbitrary direct sums, a property which can be checked on underlying $(1,1)$-categories.

Statements of the third form can be checked object-wise in the underlying $(1,1)$-categories. □

3.6.9. Write $\mathrm{open}_{\mathrm{cl}}$ (resp. $\mathrm{proper}_{\mathrm{cl}}$) for the intersection of the class open (resp. proper) with the subcategory $\mathcal{S}ch^{\mathrm{cl}} \subset \mathcal{S}ch$ of classical schemes.

Recall the following theorem of Ayoub [Ayo07], which is the classical version of [Chap. 2, Theorem 3.5.4]:

Theorem 3.6.10. *Let $(\mathbf{D}^*)^{\mathrm{cl}}$ be a motivic category of coefficients which is compactly generated*[2]*. Then $(\mathbf{D}^*)^{\mathrm{cl}}$ is $(\mathrm{open}_{\mathrm{cl}}, \mathrm{proper}_{\mathrm{cl}})$-biadjointable.*

[2]I.e., the category $\mathbf{D}(\mathrm{S})$ is compactly generated for each classical scheme S.

Proof. For noetherian (classical) bases, the claim follows from [CD09, Thms. 2.4.26 and 2.4.28], using [Chap. 2, Lemma 3.6.8] to translate.

To generalize to the case of quasi-compact quasi-separated (classical) bases, one may use the argument in the proof of [Hoy15, Prop. C.13], which clearly works *mutatis mutandis* in the setting of any compactly generated motivic category of coefficients. $\square$

3.7. Biadjointability of motivic categories of coefficients.

3.7.1. In this paragraph we will prove [Chap. 2, Theorem 3.5.4], which states that any (compactly generated) motivic category of coefficients is (open, proper)-biadjointable.

Our starting point is the result of Ayoub, [Chap. 2, Theorem 3.6.10], which says that $(\mathbf{D}^*)^{\mathrm{cl}}$ is $(\mathrm{open}_{\mathrm{cl}}, \mathrm{proper}_{\mathrm{cl}})$-biadjointable.

By the localization property, we also have that $\mathbf{D}^*$ satisfies topological invariance ([Chap. 2, Lemma 3.3.18]) and (smooth, closed)-biadjointability ([Chap. 2, Lemma 3.3.13]).

The following sequence of lemmas will then demonstrate (open, proper)-biadjointability.

3.7.2. First, we check that $(\mathrm{Adj}_{\mathrm{proper}})$ is equivalent to $(\mathrm{Adj}_{\mathrm{proper}_{\mathrm{cl}}})$:

Lemma 3.7.3. *Let $\mathbf{D}^*$ be a premotivic category of coefficients satisfying $\mathbf{S}^1$-stability, topological invariance, and weak right-adjointability along* closed. *Then $\mathbf{D}^*$ is weakly right-adjointable along a morphism f if and only if $(\mathbf{D}^*)^{\mathrm{cl}}$ is weakly right-adjointable along f_{cl}.*

Proof. Consider the commutative square

$$\begin{tikzcd} \mathrm{T}_{\mathrm{cl}} \arrow[r, "k"] \arrow[d, "f_{\mathrm{cl}}"] & \mathrm{T} \arrow[d, "f"] \\ \mathrm{S}_{\mathrm{cl}} \arrow[r, "i"] & \mathrm{S}. \end{tikzcd}$$

By topological invariance, the adjunction (k^*, k_*) is an equivalence. By weak closed-adjointability, i_* (resp. k_*) admits a right adjoint $i^!$ (resp. $k^!$).

If $(f_{\mathrm{cl}})^!$ is a right adjoint to $(f_{\mathrm{cl}})_*$, then it is easy to verify that $f^! := k_*(f_{\mathrm{cl}})^! i^!$ is a right adjoint to f_*.

Conversely if $f^!$ is a right adjoint to f_*, then $(f_{\mathrm{cl}})^! := k^! f^! i_*$ is a right adjoint to $(f_{\mathrm{cl}})_*$. $\square$

The next lemma says that $(\mathrm{BC}_{\mathrm{proper}})$ can be checked on underlying classical schemes:

Lemma 3.7.4. *Let $\mathbf{D}^*$ be an $\mathbf{S}^1$-stable premotivic category of coefficients satisfying topological invariance and right base change along* closed. *Then $\mathbf{D}^*$ satisfies right base change along a morphism g if and only if $(\mathbf{D}^*)^{\mathrm{cl}}$ satisfies right base change along g_{cl}.*

Proof. Suppose there is a cartesian square of schemes

$$\begin{tikzcd} \mathrm{T}' \arrow[r, "f'"] \arrow[d, "g'"] & \mathrm{S}' \arrow[d, "g"] \\ \mathrm{T} \arrow[r, "f"] & \mathrm{S} \end{tikzcd} \tag{3.13}$$

with g and g' proper. Consider the induced cartesian square

$$
\begin{tikzcd}
\mathrm{T}'_{\mathrm{cl}} \arrow[r, "f'_{\mathrm{cl}}"] \arrow[d, "g'_{\mathrm{cl}}"] & \mathrm{S}'_{\mathrm{cl}} \arrow[d, "g_{\mathrm{cl}}"] \\
\mathrm{T}_{\mathrm{cl}} \arrow[r, "f_{\mathrm{cl}}"] & \mathrm{S}_{\mathrm{cl}}
\end{tikzcd}
\tag{3.14}
$$

in the category of classical schemes.

We will show that the exchange 2-morphism

$$\alpha : f^* g_* \to (g')_*(f')^*$$

is invertible if and only if the exchange 2-morphism

$$\beta : (f_{\mathrm{cl}})^*(g_{\mathrm{cl}})_* \to (g'_{\mathrm{cl}})_*(f'_{\mathrm{cl}})^*$$

is invertible.

Consider the commutative cube

$$
\begin{tikzcd}
\mathrm{T}'_{\mathrm{cl}} \arrow[rr, "k'"] \arrow[dr, "g'_{\mathrm{cl}}"] \arrow[dd, "f'_{\mathrm{cl}}"'] & & \mathrm{T}' \arrow[dr, "g'"] \arrow[dd, "f'"' near start] & \\
& \mathrm{T}_{\mathrm{cl}} \arrow[rr, "k" near start] \arrow[dd, "f_{\mathrm{cl}}" near start] & & \mathrm{T} \arrow[dd, "f"] \\
\mathrm{S}'_{\mathrm{cl}} \arrow[rr, "i'" near end] \arrow[dr, "g_{\mathrm{cl}}"'] & & \mathrm{S}' \arrow[dr, "g"] & \\
& \mathrm{S}_{\mathrm{cl}} \arrow[rr, "i"] & & \mathrm{S}.
\end{tikzcd}
$$

By topological invariance, the adjunctions (i^*, i_*), (i'^*, i'_*), (k^*, k_*), (k'^*, k'_*) are all equivalences.

We claim that there is a canonical isomorphism of natural transformations

$$\beta = k^* * \alpha * (i')_*,$$

where $*$ denotes horizontal composition, from which the claim follows.

Indeed, note that there is a canonical identification $k^* f^* g_*(i')_* = k^* f^* i_*(g_{\mathrm{cl}})_* = (f_{\mathrm{cl}})^* i^* i_*(g_{\mathrm{cl}})_* = (f_{\mathrm{cl}})^*(g_{\mathrm{cl}})_*$.

On the other hand, there is a canonical identification $k^*(g')_*(f')^*(i')_* = k^*(g')_*(k')_*(f'_{\mathrm{cl}})^* = k^* k_*(g'_{\mathrm{cl}})_*(f'_{\mathrm{cl}})^* = (g'_{\mathrm{cl}})_*(f'_{\mathrm{cl}})^*$ by the closed base change formula for i'.

The fact that the induced natural transformation

$$k^* * \alpha * (i')_* : (f_{\mathrm{cl}})^*(g_{\mathrm{cl}})_* \to (g'_{\mathrm{cl}})_*(f'_{\mathrm{cl}})^*$$

coincides with α follows by careful inspection from the constructions of the exchange transformations α and β, respectively. □

The following lemma says that $(\mathrm{Proj}_{\mathrm{proper}})$ can be checked on underlying classical schemes:

Lemma 3.7.5. *Let $\mathbf{D}^*$ be an $\mathbf{S}^1$-stable premotivic category of coefficients satisfying topological invariance and the right projection formula along* closed. *Then $\mathbf{D}^*$ satisfies the right projection formula along a morphism f if and only if $(\mathbf{D}^*)^{\mathrm{cl}}$ satisfies the right projection formula along f_{cl}.*

Proof. We will show that the canonical morphism

$$\alpha(\mathcal{F}, \mathcal{G}) : f_*(\mathcal{F} \otimes_{\mathrm{T}} f^*(\mathcal{G})) \to f_*(\mathcal{F}) \otimes_{\mathrm{S}} \mathcal{G}$$

is invertible for all objects $\mathcal{F} \in \mathbf{D}(\mathrm{T})$ and $\mathcal{G} \in \mathbf{D}(\mathrm{S})$, if and only if the canonical morphism

$$\beta(\mathcal{F}_0, \mathcal{G}_0) : (f_{\mathrm{cl}})_*(\mathcal{F}_0 \otimes_{\mathrm{T}} (f_{\mathrm{cl}})^*(\mathcal{G}_0)) \to (f_{\mathrm{cl}})_*(\mathcal{F}_0) \otimes_{\mathrm{S}} \mathcal{G}_0$$

is invertible for all objects $\mathcal{F}_0 \in \mathbf{D}(\mathrm{T_{cl}})$ and $\mathcal{G}_0 \in \mathbf{D}(\mathrm{S_{cl}})$.

Consider the commutative square

$$\begin{tikzcd} \mathrm{T_{cl}} \arrow[r, "i'"] \arrow[d, "f_{\mathrm{cl}}"] & \mathrm{T} \arrow[d, "f"] \\ \mathrm{S_{cl}} \arrow[r, "i"] & \mathrm{S}. \end{tikzcd}$$

By topological invariance the adjunctions (i^*, i_*) and (k^*, k_*) are equivalences.

In particular, given objects $\mathcal{F}_0 \in \mathbf{D}(\mathrm{T_{cl}})$ and $\mathcal{G}_0 \in \mathbf{D}(\mathrm{S_{cl}})$, we can write $\mathcal{F}_0 = k^*(\mathcal{F})$ and $\mathcal{G}_0 = i^*(\mathcal{G})$, where $\mathcal{F} := k_*(\mathcal{F}_0)$ and $\mathcal{G} := i_*(\mathcal{G}_0)$, respectively. The claim follows from the fact that, for each $\mathcal{F}_0$ and $\mathcal{G}_0$, the morphism $\alpha(\mathcal{F}, \mathcal{G})$ is canonically identified with $i_*(\beta(\mathcal{F}_0, \mathcal{G}_0))$.

Indeed, note that we have $i_*((f_{\mathrm{cl}})_*(\mathcal{F}_0 \otimes (f_{\mathrm{cl}})^*(\mathcal{G}_0))) = f_*(\mathcal{F} \otimes f^*\mathcal{G})$, and $i_*((f_{\mathrm{cl}})_*(\mathcal{F}_0) \otimes \mathcal{G}_0) = i_*((f_{\mathrm{cl}})_* k^*(\mathcal{F}) \otimes i^*(\mathcal{G})) = f_*(\mathcal{F}) \otimes \mathcal{G}$ by the right projection formula for i. □

3.7.6. Finally, the following lemma says that $(\mathrm{BC}^{\mathrm{open}}_{\mathrm{proper}})$ can be checked on underlying classical schemes:

Lemma 3.7.7. *Let $\mathbf{D}^*$ be an $\mathbf{S}^1$-stable premotivic category of coefficients satisfying topological invariance and bidirectional base change along the pair* (open, closed)*. Then $\mathbf{D}^*$ satisfies bidirectional base change along* (open, proper) *if and only if $(\mathbf{D}^*)^{\mathrm{cl}}$ satisfies bidirectional base change along* $(\mathrm{open_{cl}}, \mathrm{proper_{cl}})$.

Proof. Suppose we have a cartesian square of schemes

(3.15)
$$\begin{tikzcd} \mathrm{V} \arrow[r, "f'"] \arrow[d, "j'"] & \mathrm{U} \arrow[d, "j"] \\ \mathrm{T} \arrow[r, "f"] & \mathrm{S} \end{tikzcd}$$

with f and f' proper, and j and j' open immersions, and let

(3.16)
$$\begin{tikzcd} \mathrm{V_{cl}} \arrow[r, "f'_{\mathrm{cl}}"] \arrow[d, "j'_{\mathrm{cl}}"] & \mathrm{U_{cl}} \arrow[d, "j_{\mathrm{cl}}"] \\ \mathrm{T_{cl}} \arrow[r, "f_{\mathrm{cl}}"] & \mathrm{S_{cl}} \end{tikzcd}$$

denote the induced cartesian square in the category of classical schemes.

We will show that the exchange 2-morphism

$$\alpha : j_\sharp (f')_* \to f_* (j')_\sharp$$

is invertible if and only if the exchange 2-morphism

$$\beta : (j_{\mathrm{cl}})_\sharp (f'_{\mathrm{cl}})_* \to (f_{\mathrm{cl}})_* (j'_{\mathrm{cl}})_\sharp$$

is invertible.

Consider the commutative cube

$$\begin{tikzcd}
\mathrm{V}_{\mathrm{cl}} \arrow[rr, "j'_{\mathrm{cl}}"] \arrow[dr, "k'"'] \arrow[dd, "f'_{\mathrm{cl}}"'] & & \mathrm{T}_{\mathrm{cl}} \arrow[dr, "k"] \arrow[dd, "f_{\mathrm{cl}}"] & \\
& \mathrm{V} \arrow[rr, "f'"'] \arrow[dd, "f'"] & & \mathrm{T} \arrow[dd, "f"] \\
\mathrm{U}_{\mathrm{cl}} \arrow[rr, "j_{\mathrm{cl}}"] \arrow[dr, "i'"'] & & \mathrm{S}_{\mathrm{cl}} \arrow[dr, "i"] & \\
& \mathrm{U} \arrow[rr, "j"] & & \mathrm{S}.
\end{tikzcd}$$

Using (open, closed)-base change and the fact that the adjunction $((k')^*, (k')_*)$ is an equivalence by topological invariance, one sees that there is a canonical isomorphism of 2-morphisms

$$\alpha = i_* * \beta * (k')^*,$$

from which the claim follows. $\square$

4. The formalism of six operations

4.1. Schematic correspondences.

4.1.1. Let $\mathbf{C}$ be an $(\infty,1)$-category, and let c and c' be two objects in $\mathbf{C}$.

A *correspondence* from c to c' is a diagram

$$\begin{tikzcd} d \arrow[r, "f"] \arrow[d, "g"] & c \\ c' & \end{tikzcd}$$

in $\mathbf{C}$. We will also write such a datum as a triple (d,f,g).

Let (d,f,g) and (d',f',g') be two correspondences from c to c'. A *morphism of correspondences* from (d,f,g) to (d',f',g') is a morphism $h : d \to d'$ in $\mathbf{C}$ together with a commutative diagram

$$\begin{tikzcd} d \arrow[dr, "h"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] & & \\ & d' \arrow[r, "f'"] \arrow[d, "g'"] & c \\ & c' & \end{tikzcd}$$

in $\mathbf{C}$.

From [GR16, Book-V.1], we have:

Proposition 4.1.2. *There is an $(\infty,2)$-category whose objects are objects of $\mathbf{C}$, 1-morphisms are correspondences, and 2-morphisms are morphisms of correspondences.*

4.1.3. Let $\mathcal{A}rrsch = \mathcal{A}rrows(\mathcal{S}ch)$ denote the $(\infty,1)$-category of morphisms in $\mathcal{S}ch$. To avoid ambiguity, we will use the term *schematic arrow* when we want to view a morphism of schemes as an object of $\mathcal{A}rrsch$.

Definition 4.1.4. *A* schematic correspondence *is a correspondence in the category of schematic arrows.*

We will write $\mathbf{Corrsch}$ for the $(\infty,2)$-category of schematic correspondences, and $\mathcal{C}orrsch$ for its underlying $(\infty,1)$-category.

4.1.5. Let *horiz* and *vert* be classes of morphisms in $\mathcal{A}rrsch$. Given schematic arrows f and f', a schematic correspondence (g,α,β) from f to f' is *of type* $(\mathit{horiz},\mathit{vert})$ if the morphism α (resp. β) is contained in *horiz* (resp. *vert*).

Let $\mathbf{Corrsch}_{horiz;vert}$ (resp. $\mathcal{C}orrsch_{horiz;vert}$) denote the sub-$(\infty,2)$-category of $\mathbf{Corrsch}$ (resp. sub-$(\infty,1)$-category of $\mathcal{C}orrsch$) where the 1-morphisms are spanned by correspondences of type $(\mathit{horiz},\mathit{vert})$.

Let *diag* be another class of morphisms in $\mathcal{A}rrsch$. We will write $\mathbf{Corrsch}^{diag}$ (resp. $\mathbf{Corrsch}^{diag}_{horiz;vert}$) for the sub-$(\infty,2)$-category of $\mathbf{Corrsch}$ (resp. of $\mathbf{Corrsch}_{horiz;vert}$) where the 2-morphisms are spanned by morphisms of correspondences $(g,\alpha,\beta) \to (g',\alpha',\beta')$ such that the underlying morphism of schematic arrows $\gamma : g \to g'$ is contained in *diag*.

4.1.6. Given a property of morphisms of schemes (P), we will say that a morphism of schematic arrows $\alpha : f \to g$ has the property (P) if it is of the form

$$\begin{tikzcd} \mathrm{T} \arrow[r, "f"] \arrow[d, "f"] & \mathrm{S} \arrow[d, equal] \\ \mathrm{S} \arrow[r, equal] & \mathrm{S} \end{tikzcd}$$

where the morphism f has the property (P).

For example, we may speak of morphisms of schematic arrows being *open immersions*, *proper*, *separated*, or *of finite type*. Let open (resp. proper, sep, all) denote[3] the class of open immersions (resp. proper morphisms, separated morphisms of finite type, all morphisms) of schematic arrows.

In the sequel, a special rôle will be played by the $(\infty, 2)$-category $\mathbf{Corrsch}^{\mathrm{proper}}_{\mathrm{sep:all}}$, where all denotes the class of all morphisms of schematic arrows.

4.2. The extension theorem.

4.2.1. Recall from [Chap. 0, Paragraph 3.4] the $(\infty, 2)$-category **Arenamod** of pairs $(\mathbf{O}, \mathbf{C})$, with $\mathbf{O}$ a symmetric monoidal arena and $\mathbf{C}$ an $\mathbf{O}$-module arena.

The following important theorem will be a straightforward application of the technology developed in [GR16]:

Theorem 4.2.2. *Suppose that every scheme* S *in* $\mathcal{S}ch$ *is quasi-compact and quasi-separated. Let* $\mathbf{D}^*$ *be an* (open, proper)*-biadjointable category of coefficients defined on* $\mathcal{S}ch$*. Then there exists a unique extension of* $\mathbf{D}^*$ *to a symmetric monoidal functor of* $(\infty, 2)$*-categories*

$$\mathbf{D}^*_! : \mathbf{Corrsch}^{\mathrm{proper}}_{\mathrm{sep;all}} \to (\mathbf{Arenamod})^{2\text{-op}}. \tag{4.1}$$

Combining this with [Chap. 2, Theorem 3.5.4], we obtain immediately:

Corollary 4.2.3. *Suppose that every scheme* S *in* $\mathcal{S}ch$ *is quasi-compact and quasi-separated. Let* $\mathbf{D}^*$ *be a motivic category of coefficients defined on* $\mathcal{S}ch$*. Then there exists a unique extension of* $\mathbf{D}^*$ *to a symmetric monoidal functor of* $(\infty, 2)$*-categories*

$$\mathbf{D}^*_! : \mathbf{Corrsch}^{\mathrm{proper}}_{\mathrm{sep;all}} \to (\mathbf{Arenamod})^{2\text{-op}}. \tag{4.2}$$

The rest of this section will be devoted to the proof of [Chap. 2, Theorem 4.2.2].

4.2.4. Before discussing the extension to schematic correspondences, we start by noting that we can extend any category of coefficients $\mathbf{D}^*$ to a functor of $(\infty, 1)$-categories

$$\mathbf{D}^*_{\mathcal{A}rrsch} : (\mathcal{A}rrsch)^{\mathrm{op}} \to \mathcal{A}renamod \tag{4.3}$$

defined as the composite

$$(\mathcal{A}rrsch)^{\mathrm{op}} = \mathcal{A}rrows((\mathcal{S}ch)^{\mathrm{op}}) \to \mathcal{A}rrows(\mathcal{A}renamon) \to \mathcal{A}renamod.$$

The first functor is obtained from $\mathbf{D}^* : (\mathcal{S}ch)^{\mathrm{op}} \to \mathcal{A}renamon$ by applying $\mathcal{A}rrows(-)$. The second is the canonical functor of (3.10), which sends a symmetric monoidal morphism of arenas $u : \mathbf{O} \to \mathbf{O}'$ to the pair $(\mathbf{O}, \mathbf{O}')$, where $\mathbf{O}'$ is viewed as an $\mathbf{O}$-module via u.

[3]This is a slight abuse of notation, but it will always be clear from the context whether we are referring to a class of morphisms in $\mathcal{S}ch$ or $\mathcal{A}rrsch$.

4.2.5. The functor $\mathbf{D}^*_{\mathcal{A}rrsch}$ can be described informally as follows:

On objects, it sends a schematic arrow $f : \mathrm{T} \to \mathrm{S}$ to the pair $(\mathbf{D}(\mathrm{S}), \mathbf{D}(\mathrm{T}))$, where $\mathbf{D}(\mathrm{T})$ is viewed as a $\mathbf{D}(\mathrm{S})$-module via the symmetric monoidal functor f^*.

On morphisms, it sends a morphism of schematic arrows $\alpha : g \to f$, given by a commutative square

$$\begin{tikzcd} \mathrm{T}' \arrow[r, "f'"] \arrow[d, "g'"] & \mathrm{S}' \arrow[d, "g"] \\ \mathrm{T} \arrow[r, "f"] & \mathrm{S}, \end{tikzcd}$$

to the morphism

$$\alpha^* : (\mathbf{D}(\mathrm{S}), \mathbf{D}(\mathrm{T})) \to (\mathbf{D}(\mathrm{S}'), \mathbf{D}(\mathrm{T}'))$$

defined by the pair $(g^*, (f')^*)$.

4.3. Biadjointability on schematic arrows. In this paragraph, we show that the extension of $\mathbf{D}^*$ to schematic arrows, i.e. the functor

$$\mathbf{D}^*_{\mathcal{A}rrsch} : (\mathcal{A}rrsch)^{\mathrm{op}} \to \mathcal{A}renamod,$$

still satisfies the open-base change and proper-base change properties.

Another way to say this is as follows. One can consider the evident analogues of the properties *weakly left-adjointable*, *left base change*, *weakly right-adjointable*, *right base change*, and *bidirectional base change*, for functors valued in an arbitrary 2-category. It turns out that the operation $\mathbf{D}^* \mapsto \mathbf{D}^*_{\mathcal{A}rrsch}$ preserves them, for any biadjointable $\mathbf{D}^*$.

4.3.1. The following verifies the properties of weak left-adjointability and left base change for $\mathbf{D}^*_{\mathcal{A}rrsch}$:

Lemma 4.3.2. *(i) For every open immersion $\beta : j \to \mathrm{id}_\mathrm{S}$ of schematic arrows given by a commutative square*

$$\begin{tikzcd} \mathrm{T} \arrow[r, "j"] \arrow[d, "j"] & \mathrm{S} \arrow[d, equal] \\ \mathrm{S} \arrow[r, equal] & \mathrm{S}, \end{tikzcd}$$

the induced morphism

$$\beta^* := \mathbf{D}^*_{\mathcal{A}rrsch}(\beta) : (\mathbf{D}(\mathrm{S}), \mathbf{D}(\mathrm{S})) \to (\mathbf{D}(\mathrm{S}), \mathbf{D}(\mathrm{T}))$$

admits a left adjoint $\beta_\sharp$ in the 2-category **Arenamod**.

(ii) For every cartesian square of schematic arrows

$$\begin{tikzcd} j' \arrow[r, "\alpha'"] \arrow[d, "\beta'"] & j \arrow[d, "\beta"] \\ \mathrm{id}_{\mathrm{S}'} \arrow[r, "\alpha"] & \mathrm{id}_\mathrm{S} \end{tikzcd} \tag{4.4}$$

with β and β' open immersions, the induced commutative square in **Arenamod**

$$\begin{tikzcd} \mathbf{D}^*_{\mathcal{A}rrsch}(f) \arrow[r, "\alpha^*"] \arrow[d, "\beta^*"] & \mathbf{D}^*_{\mathcal{A}rrsch}(f') \arrow[d, "(\beta')^*"] \\ \mathbf{D}^*_{\mathcal{A}rrsch}(j) \arrow[r, "(\alpha')^*"] & \mathbf{D}^*_{\mathcal{A}rrsch}(j'), \end{tikzcd}$$

obtained by applying the functor $\mathbf{D}^*_{\mathcal{A}rrsch}$, is vertically left-adjointable.

Dually, we have:

Lemma 4.3.3. *(i) For every proper morphism $\alpha : f \to \mathrm{id}_{\mathrm{S}}$ of schematic arrows, given by a commutative square*

$$\begin{tikzcd} \mathrm{T} \arrow[r, "f"] \arrow[d, "f"] & \mathrm{S} \arrow[d, equal] \\ \mathrm{S} \arrow[r, equal] & \mathrm{S}, \end{tikzcd}$$

the induced morphism

$$\alpha^* : (\mathbf{D}(\mathrm{S}), \mathbf{D}(\mathrm{S})) \to (\mathbf{D}(\mathrm{S}), \mathbf{D}(\mathrm{T}))$$

admits a right adjoint α_* in the 2-category **Arenamod**.

(ii) For every cartesian square of schematic arrows

$$\begin{tikzcd} g' \arrow[r, "\alpha'"] \arrow[d, "\beta'"] & g \arrow[d, "\beta"] \\ f' \arrow[r, "\alpha"] & f, \end{tikzcd}$$

with α and α' proper, the induced commutative square in **Arenamod**

$$\begin{tikzcd} \mathbf{D}^*_{\mathcal{A}rrsch}(f) \arrow[r, "\alpha^*"] \arrow[d, "\beta^*"] & \mathbf{D}^*_{\mathcal{A}rrsch}(f') \arrow[d, "(\beta')^*"] \\ \mathbf{D}^*_{\mathcal{A}rrsch}(g) \arrow[r, "(\alpha')^*"] & \mathbf{D}^*_{\mathcal{A}rrsch}(g'), \end{tikzcd}$$

obtained by applying the functor $\mathbf{D}^*_{\mathcal{A}rrsch}$, is horizontally right-adjointable.

Finally, we have the bidirectional base change property:

Lemma 4.3.4. *For all cartesian squares of schematic arrows*

$$\begin{tikzcd} g' \arrow[r, "\alpha'"] \arrow[d, "\beta'"] & f' \arrow[d, "\beta"] \\ g \arrow[r, "\alpha"] & f \end{tikzcd}$$

with β and β' open immersions, and α and α' proper, the induced commutative square in $\mathcal{A}renamod$

$$\begin{tikzcd} (\mathbf{D}^*, \mathbf{D}^*)(f') \arrow[r, "(q')^*"] \arrow[d, "p_\sharp"] & (\mathbf{D}^*, \mathbf{D}^*)(g') \arrow[d, "p'_\sharp"] \\ (\mathbf{D}^*, \mathbf{D}^*)(f) \arrow[r, "q^*"] & (\mathbf{D}^*, \mathbf{D}^*)(g'), \end{tikzcd}$$

obtained by applying the functor $\mathbf{D}^*_{\mathcal{A}rrsch}$ and vertically passing to left adjoints, is horizontally right-adjointable.

4.3.5. All three of these lemmas are purely formal exercises. We will only prove the first statement; the second is completely dual, and the third is also similar.

Let $\beta : j \to f$ be an open immersion given by a commutative square as above. Recall that the morphism

$$\beta^* : (\mathbf{D}(\mathrm{S}), \mathbf{D}(\mathrm{S})) \to (\mathbf{D}(\mathrm{S}), \mathbf{D}(\mathrm{T}))$$

is defined by the pair $(\mathrm{id}_{\mathbf{D}(\mathrm{S})}, j^*)$.

We define a morphism in **Arenamod**

$$\beta_\sharp : (\mathbf{D}(\mathrm{S}), \mathbf{D}(\mathrm{T})) \to (\mathbf{D}(\mathrm{S}), \mathbf{D}(\mathrm{S}))$$

given by the triple $(\mathrm{id}_{\mathbf{D}(\mathrm{S})}, j_\sharp)$.

We will show that $\beta_\sharp$ is left adjoint to β^*, by constructing unit and counit morphisms satisfying the triangle identities.

4.3.6. Let us define the unit 2-morphism $\eta : \mathrm{id}_{(\mathbf{D}(\mathrm{S}),\mathbf{D}(\mathrm{T}))} \to \beta^*\beta_\sharp$ in **Arenamod**.

This is the 2-morphism

$$\eta : (\mathrm{id}_{\mathbf{D}(\mathrm{S})}, \mathrm{id}_{\mathbf{D}(\mathrm{T})}) \to (\mathrm{id}_{\mathbf{D}(\mathrm{S})}, j^*j_\sharp)$$

by the data:

- The natural transformation $\mathrm{id}_{\mathbf{D}(\mathrm{S})} \to \mathrm{id}_{\mathbf{D}(\mathrm{S})}$ given by the identity.
- The $\mathbf{D}(\mathrm{S})$-linear natural transformation $\mathrm{id}_{\mathbf{D}(\mathrm{T})} \to j^*j_\sharp$ given by the unit of the adjunction $(j_\sharp, j^*)$.

4.3.7. Let us define the counit 2-morphism $\varepsilon : \beta_\sharp\beta^* \to \mathrm{id}_{(\mathbf{D}(\mathrm{S}),\mathbf{D}(\mathrm{S}))}$ in **Arenamod**.

This is the 2-morphism in **Arenamod**

$$\varepsilon : (\mathrm{id}_{\mathbf{D}(\mathrm{S})}, j_\sharp j^*) \to (\mathrm{id}_{\mathbf{D}(\mathrm{S})}, \mathrm{id}_{\mathbf{D}(\mathrm{S})})$$

by the data:

- The natural transformation $\mathrm{id}_{\mathbf{D}(\mathrm{S})} \to \mathrm{id}_{\mathbf{D}(\mathrm{S})}$ given by the identity.
- The $\mathbf{D}(\mathrm{S})$-linear natural transformation $j_\sharp j^* \to \mathrm{id}_{\mathbf{D}(\mathrm{S})}$ given by the counit of the adjunction $(j_\sharp, j^*)$.

4.3.8. It is immediate that the data of the 2-morphisms η and ε defines an adjunction $(\beta_\sharp, \beta^*)$ in the 2-category **Arenamod**.

4.3.9. We next verify the base change property, i.e. statement (ii) of [Chap. 2, Lemma 4.3.2].

Let $\beta : j \to \mathrm{id}_{\mathbf{D}(\mathrm{S})}$ and $\beta' : j' \to \mathrm{id}_{\mathbf{D}(\mathrm{S}')}$ be given by commutative squares

$$\begin{tikzcd} \mathrm{T} \arrow[r, "j"] \arrow[d, "j"] & \mathrm{S} \arrow[d, equal] \\ \mathrm{S} \arrow[r, equal] & \mathrm{S}, \end{tikzcd} \qquad \begin{tikzcd} \mathrm{T}' \arrow[r, "j'"] \arrow[d, "j'"] & \mathrm{S}' \arrow[d, equal] \\ \mathrm{S}' \arrow[r, equal] & \mathrm{S}', \end{tikzcd}$$

respectively.

Let $\alpha : \mathrm{id}_{\mathbf{D}(\mathrm{S}')} \to \mathrm{id}_{\mathbf{D}(\mathrm{S})}$ and $\alpha' : j' \to j$ be given by commutative squares

$$\begin{tikzcd} \mathrm{S}' \arrow[r, equal] \arrow[d, "f"] & \mathrm{S}' \arrow[d, "f"] \\ \mathrm{S} \arrow[r, equal] & \mathrm{S}, \end{tikzcd} \qquad \begin{tikzcd} \mathrm{T}' \arrow[r, "j'"] \arrow[d, "g"] & \mathrm{S}' \arrow[d, "f"] \\ \mathrm{T} \arrow[r, "j"] & \mathrm{S}, \end{tikzcd}$$

respectively.

Assuming that the square (4.4) is cartesian, we have to show that the exchange 2-morphism

$$(\beta')_\sharp(\alpha')^* \to \alpha^*\beta_\sharp \tag{4.5}$$

is invertible.

Note that (4.4) induces a cartesian square of schemes

$$\begin{tikzcd} \mathrm{T}' \arrow[r, "j'"] \arrow[d, "g"] & \mathrm{S}' \arrow[d, "f"] \\ \mathrm{T} \arrow[r, "j"] & \mathrm{S}. \end{tikzcd} \tag{4.6}$$

4.3.10. The composite $\alpha^*\beta_\sharp$ is given by the pair $(f^*, f^*j_\sharp)$.

The composite $(\beta')_\sharp(\alpha')^*$ is given by the pair $(f^*, (j')_\sharp g^*)$.

The 2-morphism (4.5) is given by the following data:

- The natural transformation $f^* \to f^*$ given by the identity.
- The $\mathbf{D}(\mathrm{S})$-linear transformation $f^*j_\sharp \to (j')_\sharp g^*$ given by the exchange transformation associated to the cartesian square (4.6).

By the axiom ($\mathrm{BC}^{\mathrm{open}}$) it follows that this 2-morphism is invertible.

4.4. The extension theorem, proof. In this paragraph we will prove [Chap. 2, Theorem 4.2.2].

4.4.1. We begin with the functor

$$\mathbf{D}^*_{\mathcal{A}rrsch} : (\mathcal{A}rrsch)^{\mathrm{op}} \to \mathcal{A}renamod$$

defined in (4.3).

The first part of the construction of the 2-functor $\mathbf{D}^*_!$ (4.2) consists of the extension of $\mathbf{D}^*_{\mathcal{A}rrsch}$ to a symmetric monoidal functor of $(\infty, 2)$-categories

$$\mathbf{D}^*_\sharp : \mathbf{Corrsch}^{\mathrm{iso}}_{\mathrm{open;all}} \to \mathbf{Arenamod}. \tag{4.7}$$

We will obtain this extension by using [GR16, Book-V.1, Thm. 3.2.2(b)], which applies because of the following two observations:

(1) The triple (all, open, open) satisfies the conditions of [GR16, Book-V.1, 1.1.1]. That is to say, the class of open immersions of schematic arrows is stable under base change and satisfies the 2-of-3 property. This follows immediately from the respective properties for the class of open immersions of schemes.

(2) The functor $\mathbf{D}^*_{\mathcal{A}rrsch}$ satisfies the right Beck-Chevalley condition [GR16, Book-V.1, Def. 3.1.5] with respect to the class of open immersions. This is the content of [Chap. 2, Lemma 4.3.2].

4.4.2. Next, we want to extend $\mathbf{D}^*_\sharp$ (4.7) to a symmetric monoidal functor of $(\infty, 2)$-categories

$$\mathbf{D}^*_! : \mathbf{Corrsch}^{\mathrm{proper}}_{\mathrm{sep;all}} \to (\mathbf{Arenamod})^{2\text{-op}}. \tag{4.8}$$

For this, we flip the directions of the 1- and 2-morphisms in $\mathbf{D}^*_\sharp$ to obtain a 2-functor

$$\mathbf{Corrsch}^{\mathrm{iso}}_{\mathrm{all;open}} \to (\mathbf{Arenamod})^{1\&2\text{-op}}. \tag{4.9}$$

In order to apply [GR16, Book V.1, Thm. 5.2.4], we need to check the following:

(1) The classes proper and sep satisfy the conditions of [GR16, Book-V.1, 1.1.1 and 5.1.1]. That is to say, the class proper is stable under base change, and the class sep is stable under base change and satisfies the 2-of-3 property.

(2) The classes open and proper satisfy the condition of [GR16, Book-V.1, 5.1.2]. That is to say, every morphism in the intersection of the classes open and proper is a monomorphism. This follows immediately from the corresponding fact for the classes open and proper in $\mathcal{S}ch$. In fact, recall that every open immersion is a monomorphism.

(3) The classes proper, open and all satisfy the condition of [GR16, Book-V.1, 5.1.3]. That is to say, for every morphism γ of schematic arrows which is separated and of finite type, the

category of commutative diagrams

$$\begin{tikzcd} f \arrow[rr, "\gamma"] \arrow[dr, "\beta"'] & & g, \\ & f' \arrow[ur, "\alpha"'] & \end{tikzcd}$$

with β an open immersion and α proper, is contractible. This follows immediately from the corresponding statement for morphisms of schemes, which is [Chap. 0, Proposition 6.3.4].

(4) The functor $\mathbf{D}^*_{\mathcal{A}rrsch}$ satisfies the left Beck-Chevalley condition of [GR16, Book-V.1, Def. 3.1.2] with respect to the class of proper morphisms. This is the content of [Chap. 2, Lemma 4.3.3].

(5) The functor (4.9) satisfies the condition of [GR16, Book V.1, 5.2.2]. This is the content of [Chap. 2, Lemma 4.3.4].

Hence we may apply [GR16, Thm. 5.2.4] to obtain a 2-functor

$$\mathbf{Corrsch}^{\text{proper}}_{\text{all;sep}} \to (\mathbf{Arenamod})^{1\&2\text{-op}}.$$

By flipping the directions of 1-morphisms we obtain the 2-functor

$$\mathbf{D}^*_! : \mathbf{Corrsch}^{\text{proper}}_{\text{sep;all}} \to (\mathbf{Arenamod})^{2\text{-op}}$$

which is the extension desired.

5. Example: the stable motivic homotopy category

5.1. The functor $\mathcal{SH}^*$. In this paragraph we will define the stable motivic homotopy category, introduced in Chapter 1, as a category of coefficients in the sense of [Chap. 2, Sect. 2].

Here the notation $\mathcal{S}ch$ will be used for the category of quasi-compact quasi-separated schemes.

5.1.1. Let $\mathbf{Corrsch}^{\text{proper}}_{\text{sep;all}}$ denote the $(\infty, 2)$-category of schematic correspondences (see [Chap. 2, Paragraph 4.1]). In this section we will show:

Theorem 5.1.2. *The assignment* $\mathrm{S} \mapsto \mathcal{SH}(\mathrm{S})$ *lifts to a symmetric monoidal functor of* $(\infty, 2)$*-categories*

$$\mathcal{SH}^*_! : \mathbf{Corrsch}^{\text{proper}}_{\text{sep;all}} \to (\mathbf{Arenamod})^{\text{2-op}}. \tag{5.1}$$

5.1.3. We begin by lifting the assignment $\mathrm{S} \mapsto \mathcal{SH}(\mathrm{S})$ to a motivic category of coefficients.

Consider the canonical functor

$$\mathcal{A}rrows(\mathcal{S}ch) \to \mathcal{S}ch$$

sending a morphism of schemes $f : \mathrm{T} \to \mathrm{S}$ to its target S.

By abstract nonsense (see [GR16, Chap. I.1]), this is a cartesian fibration which corresponds, via straightening/unstraightening, to the presheaf of categories $(\mathcal{S}ch)^{\text{op}} \to (\infty, 1)\text{-}\mathcal{C}at$ given object-wise by the assignment $\mathrm{S} \mapsto \mathcal{S}ch_{/\mathrm{S}}$.

5.1.4. Let $\mathcal{S}m$ denote the full subcategory of $\mathcal{A}rrows(\mathcal{S}ch)$ spanned by smooth morphisms of schemes. Since smoothness is stable under base change, it is easily verified that the composite

$$\mathcal{S}m \hookrightarrow \mathcal{A}rrows(\mathcal{S}ch) \to \mathcal{S}ch$$

is also a cartesian fibration.

This corresponds by straightening/unstraightening to a presheaf of categories $(\mathcal{S}ch)^{\text{op}} \to (\infty, 1)\text{-}\mathcal{C}at$ given object-wise by the assignment

$$\mathrm{S} \mapsto \mathcal{S}m_{/\mathrm{S}}.$$

5.1.5. Applying the canonical functor $\mathbf{C} \mapsto \mathcal{P}(\mathbf{C})$ object-wise, we obtain a presheaf of arenas $(\mathcal{S}ch)^{\text{op}} \to \mathcal{A}rena$ given on objects by the assignment

$$\mathrm{S} \mapsto \mathcal{S}pc(\mathrm{S})$$

and on morphisms by $f \mapsto f^*$.

Note that this lifts to a presheaf of *symmetric monoidal* arenas: each category $\mathcal{S}pc(\mathrm{S})$ is cartesian monoidal, and the morphisms f^* are symmetric monoidal (i.e. commute with finite products).[4]

Hence we have a functor

$$(\mathcal{S}ch)^{\text{op}} \to \mathcal{A}renamon.$$

[4]Note that in the cartesian monoidal case, symmetric monoidality is a *property* of functors, not a structure.

5.1.6. Consider the category of pairs $(\mathbf{C}, \mathrm{S})$, with $\mathbf{C}$ an arena and S an essentially small set of morphisms; see [Chap. 0, Paragraph 2.6].

Since the classes of Nisnevich covers and $\mathbf{A}^1$-projections are stable under base change, respectively, one sees that the functor $(\mathcal{S}ch)^{\mathrm{op}} \to \mathcal{A}renamon$ lifts to (commutative monoids in) this category of pairs, where we send a scheme S to the category $\mathcal{S}pc(\mathrm{S})$ together with the small set of Nisnevich-local and $\mathbf{A}^1$-local isomorphisms.

Applying the symmetric monoidal functor given by object-wise localization of pairs (see *loc. cit.*), we obtain a functor $(\mathcal{S}ch)^{\mathrm{op}} \to \mathcal{A}renamon$ given on object by the assignment

$$\mathrm{S} \mapsto \mathcal{H}(\mathrm{S}).$$

5.1.7. Applying the functor $\mathbf{C} \mapsto \mathbf{C}_\bullet$ object-wise, we obtain a functor $(\mathcal{S}ch)^{\mathrm{op}} \to \mathcal{A}renamon$ given on objects by the assignment

$$\mathrm{S} \mapsto \mathcal{H}(\mathrm{S})_\bullet.$$

5.1.8. Fix a family of pointed fibred spaces $(\mathcal{T}_\mathrm{S})_\mathrm{S}$ as in [Chap. 1, Paragraph 5.3]. The properties of the "formal inversion" $\mathcal{H}(\mathrm{S})_\bullet \mapsto \mathcal{SH}_\mathcal{T}(\mathrm{S})$ studied in [Rob14] provide a functor

$$(\mathcal{S}ch)^{\mathrm{op}} \to \mathcal{A}renamod,$$

or equivalently a symmetric monoidal functor

$$(\mathcal{S}ch)^{\mathrm{op}} \to \mathcal{A}rena,$$

given on objects by the assignment

$$\mathrm{S} \mapsto \mathcal{SH}_\mathcal{T}(\mathrm{S}).$$

See especially §9.1 of *loc. cit.*.

This is the category of coefficients desired.

5.2. The 2-functor $\mathcal{SH}^*_!$. In this paragraph we obtain the 2-functor $\mathcal{SH}^*_!$ encoding the formalism of six operations on motivic spectra.

5.2.1. First, we have that the category of coefficients $\mathcal{SH}^*$ is left-adjointable along smooth morphisms. This was demonstrated in [Chap. 1, Sect. 6].

Hence $\mathcal{SH}^*$ is premotivic.

5.2.2. We have homotopy invariance for $\mathcal{SH}^*$ by construction, using point (iii) of [Chap. 2, Lemma 3.2.4].

5.2.3. We now fix the family $(\mathcal{T}_\mathrm{S})_\mathrm{S} = (\mathbf{P}^1_\mathrm{S})_\mathrm{S}$, and write $\mathcal{SH}(\mathrm{S}) = \mathcal{SH}_{\mathbf{P}^1}(\mathrm{S})$. Recall that $\mathbf{P}^1_\mathrm{S}$ denotes, by abuse of notation, the pointed motivic space $(\mathrm{M}_\mathrm{S}(\mathbf{P}^1_\mathrm{S}), \infty)$.

Recall also that there is a canonical identification $\mathbf{P}^1_\mathrm{S} = \mathrm{Th}_\mathrm{S}(\mathbf{A}^1_\mathrm{S})$ for each S (see [Chap. 1, Corollary 8.2.4]). Hence we have Thom stability for $\mathcal{SH}^*$ by construction, using point (iv) of [Chap. 2, Corollary 3.4.19].

5.2.4. We have the localization property for $\mathcal{SH}^*$; this was the main result of Chapter 1 (see [Chap. 1, Paragraph 7.4]).

5.2.5. In summary, the category of coefficients $\mathcal{SH}^*$ is *motivic*. By [Chap. 2, Corollary 4.2.3], we obtain the symmetric monoidal functor of $(\infty, 2)$-categories

$$\mathcal{SH}^*_! : \mathbf{Corrsch}^{\mathrm{proper}}_{\mathrm{sep;all}} \to (\mathbf{Arenamod})^{2\text{-op}}. \tag{5.2}$$

desired, a unique extension of $\mathcal{SH}^*$, encoding the formalism of six operations on motivic spectra.

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